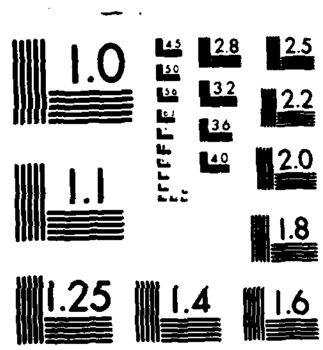
AD-A127 358 ON THE DISTRIBUTION OF THE LIKELIHOOD RATIO CRITERIA 1/1
ASSOCIATED WITH K SA..(U) AIR FORCE INST OF TECH
WRIGHT-PATTERSON AFB OH SCHOOL OF ENGI.. A J SHERWOOD
UNCLASSIFIED MAR 83 AFIT/GAM/MA/83M-1 F/G 12/1 NL

END
DATE
FILMED
5 -83
DTIC


1.0
2.8
2.5
3.2
2.2
3.6
4.0
2.0
1.1
1.8
1.25
1.4
1.6

A127358

ON THE DISTRIBUTION OF THE LIKELIHOOD RATIO CRITERIA ASSOCIATED WITH K SAMPLES OF TWO CORRELATED RANDOM VARIABLES

THESIS

AFIT/GAM/MA/83M-1

Arthur J. Sherwood
Capt USAF

DEPARTMENT OF THE AIR FORCE
AIR UNIVERSITY (ATC)

AIR FORCE INSTITUTE OF TECHNOLOGY

Wright-Patterson Air Force Base, Ohio

83 04 28 089

AFIT/GAM/MA/83M-1

1

ON THE DISTRIBUTION OF THE LIKELIHOOD RATIO CRITERIA ASSOCIATED WITH K SAMPLES OF TWO CORRELATED RANDOM VARIABLES

THESIS

AFIT/GAM/MA/83M-1 Arthur J. Sherwood
Capt USAF

AFIT/GAM/MA/83M-1

ON THE DISTRIBUTION OF THE LIKELIHOOD RATIO CRITERIA ASSOCIATED WITH K SAMPLES OF TWO CORRELATED RANDOM VARIABLES

THESIS

Presented to the Faculty of the School of Engineering
of the Air Force Institute of Technology
Air University
in Partial Fulfillment of the
Requirements for the Degree of
Master of Science

by

Arthur J. Sherwood, B.S.
Capt USAF

Graduate Applied Mathematics

March 1983

Preface

Methods employed when testing hypotheses concerning populations must consider the possibility of correlation existing among the variables that characterize the entity being tested. Multivariate Analysis provides a method to effectively deal with this correlation problem.

This report uses Multivariate Theory to derive the sampling distributions of the likelihood ratio criteria for two correlated variables and k populations. In this endeavor I offer my deepest gratitude to my advisor, Dr. B. N. Nagarsenker, not only for his guidance throughout this project, but for teaching me the Multivariate Theory upon which the entire thesis is based.

I would also like to thank my reader, Lt Col Richard Kulp, whose advice and editing aided me greatly; Mr. Jerry Petrak and the Engineering and Design Data Group, AFWAL/MLSE, for supplying the experimental data and assisting in its analysis; and my typist, Phyllis Reynolds, for her superb job.

Finally, I want to express a special thanks to my wife, Lois, for her dedication and support; and our children, Robert, Daniel, and Cristina, who were a constant blessing throughout the entire AFIT program.

— Arthur J. Sherwood

Contents

		Page
Preface		ii
Nomenclature		v
List of Tables		vi
Abstract		vii
I.	Introduction	1
	Background	2
	The Likelihood Ratio Criterion of Neyman and Pearson	3
	Objective	8
II.	Statistical Preliminaries	10
	Multivariate Normal Distribution	10
	Maximum Likelihood Estimates (MLE) of $\underline{\mu}$ and $\underline{\Sigma}$.	12
	The Wishart Distribution	15
	Theorems Concerning the Wishart Distribution.	16
	Maximum Likelihood Ratio Test	18
III.	Derivation of the Criteria	20
	Likelihood Estimates of $\underline{\mu}^g$ and $\underline{\Sigma}_g$	21
	Derivation of Criteria for H_0	23
	Derivation of Criteria for H_1	26
	Derivation of Criteria for H_2	27
	Particular Cases	28
IV.	Derivation of the Moments	30
	The h^{th} Moment of λ_0	30
	The h^{th} Moment of λ_1	34
	The h^{th} Moment of λ_2	36
V.	Distribution of the Criteria	39
	The Distribution of λ_0	39
	The Distribution of λ_1	44
	The Distribution of λ_2	48
	Numerical Computations	49
	Summary	50

		Page
VI.	Asymptotic Approximation to the Distribution . .	52
	Asymptotic Expansion of the Distribution λ_0. .	52
	Asymptotic Expansion of the Distribution λ_1. .	58
VII.	Practical Illustration	64
VIII.	Conclusion	68
Bibliography	. . . .	70
Appendix:	Tables	72
Vita	. . . .	83

Nomenclature

1. $\underline{A}$: Matrix of sums of squares and sums of products
2. $\underline{A}'$: Transpose of $\underline{A}$
3. $\underline{A}^{-1}$: Inverse of $\underline{A}$
4. $|\underline{A}|$: Determinate of $\underline{A}$
5. c.d.f. : Cumulative distribution function
6. E : Expectation Operator
7. exp : Exponent
8. log : Logarithm to the base e
9. MLE : Maximum Likelihood Estimate
10. p.d.f. : Probability density function
11. $\underline{A} > 0$: $\underline{A}$ is positive definite
12. tr : trace
13. $\underline{\mu}^g$: mean of the g^{th} population
14. $\underline{\bar{x}}^g$: sample mean of the g^{th} population
15. Ω : Parameter space of the null hypothesis
16. ω : Parameter space of alternate hypothesis
17. $\underline{\Sigma}_g$: Variance-covariance matrix of the g^{th} population

List of Tables

Table		Page
5-1.	Evaluation of c.d.f. of Theorem 5.2 for m Terms	50
6-1.	Comparison of Approximate and Exact Distribution, $\alpha = .01$	56
6-2.	Comparison of Approximate and Exact Distribution, $\alpha = .05$	57
6-3.	Comparison of Approximate and Exact Distribution, $\alpha = .01$	62
6-4.	Comparison of Approximate and Exact Distribution, $\alpha = .05$	63
7-1.	Tensile Strength: Variables L and T	65
7-2.	Summary of Calculations	65
I.	Percentage Points of L_0	73
II.	Percentage Points of L_1	75
III.	Percentage Points of L_2	77
IV.	Percentage Points of ω_0	79
V.	Percentage Points of ω_1	82

Abstract

Let a random sample of size N_g be drawn from a p-variate normal population $N_p(\underline{\mu}^g, \underline{\Sigma}_g)$ $g = (1,2,\ldots,k)$. In this thesis we consider the problem of testing the following hypotheses:

[i] H_0: $\underline{\mu}^1 = \underline{\mu}^2 = \ldots = \underline{\mu}^k$,

$\underline{\Sigma}_1 = \underline{\Sigma}_2 = \ldots = \underline{\Sigma}_k$

[ii] H_1: $\underline{\Sigma}_1 = \underline{\Sigma}_2 = \ldots = \underline{\Sigma}_k$. The means can be any value

[iii] H_2: $\underline{\mu}^1 = \underline{\mu}^2 = \underline{\mu}^k$ given $\underline{\Sigma}_1 = \underline{\Sigma}_2 = \ldots = \underline{\Sigma}_k$

against the general alternatives.

Likelihood ratio criteria and their sampling distributions are derived for p = 2 and equal sample sizes. From these distributions, tables of percentage points for the three likelihood ratio criteria are computed.

A useful approximation is also obtained. The theoretical results are then applied to actual data.

ON THE DISTRIBUTION OF THE LIKELIHOOD RATIO CRITERIA ASSOCIATED WITH K SAMPLES OF TWO CORRELATED RANDOM VARIABLES

I. Introduction

Statistical techniques enable experimenters to analyze the variation and covariation that exists between the measured characteristic of observed events. Analysts seek to assign causes to this variation, test and compare alternative hypothesis and express the results in terms of a measure of probability. Some of these hypotheses are: (1) Is the sample from a specified population? (2) Are the k samples from a common but unspecified population? (3) Is the population completely specified or only partly? (4) Do several populations with different means have the same standard deviations? (5) Are the variables being tested correlated?

One approach, the Analysis of Variance, developed by R. A. Fisher, is based on the assumption that the unexplained variation (residuals) is normally independently distributed and the populations have the same standard deviation. The assumption that the standard deviations are the same is not always true and, therefore, the results

obtained could be misleading to the user of the information. Multivariate an lysis theory is well suited in this case, specifically where two or more correlated variables are involved.

Background

In early developments of hypothesis testing, the fundamental hypothesis, H: Are the two samples X_1 and X_2, from the same unknown normal population k, was treated by Professor V. Romanovsky in his paper entitled "On Criteria that Two Given Samples Belong to the Same Normal Population" (Ref 14). Romanovsky approached the problem assuming the hypothesis H to be true and derived the distribution function for his test criteria. He provided four alternative criteria for testing his hypothesis H. These criteria are as follows:

$$\alpha = \frac{(\bar{x}_1 - \bar{x}_2)^2}{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \tag{1.1}$$

$$\mu = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} \tag{1.2}$$

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{n_1 s_1^2 + n_2 s_2^2}} \sqrt{\frac{n_1 n_2 (n_1 + n_2 - 2)}{n_1 + n_2}} \tag{1.3}$$

$$\theta = \frac{s_2^2}{s_1^2} \qquad (1.4)$$

Neyman and Pearson (Ref 11:201) point out that the criterion α, μ, and t are not sensitive to differences in population standard deviations. For example, the pairs of samples may have s_1 and s_2 almost equal, whereas a second pair, s_1 and s_2 could vary greatly, yet the value of t may be the same in both cases. The criteria θ, does distinguish between the population standard deviations, but is not sensitive to the difference between their means. Because of the restriction on these criterion, further research is necessary to derive a test statistic. The test statistic should have the following properties: (1) be able to distinguish between population standard deviations and between their means, and (2) the test statistic should be selected in such a manner that it will minimize the danger of accepting a false hypothesis.

Neyman and Pearson use the likelihood ratio test to derive a test statistic for one variable and two populations that satisfies the above requirements.

The Likelihood Ratio Criterion of Neyman and Pearson

R. A. Fisher in the early 1920s proposed a general method of estimation called the method of maximum likelihood from which the likelihood ratio criteria for testing

hypotheses was developed. The method produced sufficient estimates for the parameters whenever they existed and the estimates are asymptotically, $(n \to \infty)$, minimum variance unbiased estimators.

Now we will discuss the likelihood ratio criterion of Neyman and Pearson.

Let the stochastically independent random variables X_1 and X_2 be chosen from some normal populations k_1 and k_2 where the means and variances are any values. Then our parameter space $\Omega = (\mu^1, \mu^2, \Sigma_1, \Sigma_2)$, where $(-\infty < \mu^1 < \infty)$, $(-\infty < \mu^2 < \infty)$, $(0 < \Sigma_1 < \infty)$, $(0 < \Sigma_2 < \infty)$. We wish to test the hypothesis H_0: $\mu^1 = \mu^2$, $\Sigma_1 = \Sigma_2$ against all alternatives. Under H_0 let ω be such that $(-\infty < \mu^1 = \mu^2 < \infty)$ and $(0 < \Sigma_1 = \Sigma_2 < \infty)$. Let $L(\Omega)$ and $L(\omega)$ define the likelihood function for Ω and ω respectively and $L(\hat{\Omega})$ and $L(\hat{\omega})$ be their maxima. Then Neyman and Pearson obtained the likelihood ratio criteria for testing H_0 in the form (Ref 11:103):

$$\lambda_H = \frac{L(\hat{\omega})}{L(\hat{\Omega})} = \left[\frac{s_1}{s_0}\right]^{n_1}\left[\frac{s_2}{s_0}\right]^{n_2}$$

where n_1 is the sample size for X_1,

n_2 is the sample size for X_2,

s_1 is the standard deviation for X_1,

s_2 is the standard deviation for X_2, and

s_0 is the standard deviation obtained by combining the n_1 and n_2 variables of the samples X_1 and X_2.

Our criteria λ_{H_0} lies between 0 and 1. If our hypothesis H_0 is true we would expect the ratio of $L(\hat{\omega})$ to $L(\hat{\Omega})$ to approach unity. The closer to unity the more confidence we have that H_0 is true. However, if λ_{H_0} approaches zero we become more certain that the hypothesis H_0 is false.

The nature of the hypothesis H_0 allows us to separate it into two hypotheses: (1) H_1: The samples come from unknown populations with the same variance, but with means having any value whatever; and (2) H_2: The means are the same, assuring equal variances and normal populations.

If we use μ and t from equations (1.2) and (1.3) and the equation

$$s_0^2 = \frac{(n_1 s_1^2 + n_2 s_2^2)}{(n_1 + n_2)} + \frac{n_1 n_2}{(n_1 + n_2)^2} (\bar{x}_1 - \bar{x}_2)^2$$

then λ_H can be represented as a function of Romanovsky's criteria t and θ from equations (1.3) and (1.4).

$$\lambda = (n_1 + n_2)^{\frac{(n_1 + n_2)}{2}} \frac{n_2}{\theta^2} (n_1 + n_2\theta)^{-\frac{(n_1 + n_2)}{2}}$$

$$\left(1 + \frac{t^2}{n_1 + n_2 - 2}\right)^{-\frac{(n_1 + n_2)}{2}} \qquad (1.5)$$

From equation (1.5) Neyman and Pearson (Ref 11:104) derived the likelihood function criteria for testing H_1 and H_2. Thus, the likelihood of H_1 is

$$\lambda_{H_1} = (n_1 + n_2)^{\frac{(n_1 + n_2)}{2}} \frac{n_2}{\theta^2} (n_1 + n_2\theta)^{-\frac{(n_1 + n_2)}{2}} \tag{1.6}$$

The likelihood of H_2 is

$$\lambda_{H_2} = \left(1 + \frac{t^2}{n_1 + n_2 - 2}\right)^{-\frac{(n_1 + n_2)}{2}} \tag{1.7}$$

Combining λ_{H_1} and λ_{H_2} the results are

$$\lambda = \lambda_{H_1} \lambda_{H_2} \tag{1.8}$$

From equation (1.5) it can be seen that λ_H obtains its maximum value of unity when both $\theta = 1$ and $t = 0$, or $s_1 = s_2$ and $\bar{x}_1 = \bar{x}_2$. λ_H will decrease towards zero when

a) $\theta \to 0$ or s_2 becomes small compared with s_1.

b) $\theta \to \infty$ or s_1 becomes small compared with s_2.

c) $|t| \to \infty$ or $|\bar{x}_1 - \bar{x}_2|$ increases compared with

$$V = \sqrt{\frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$

Thus, even if $\bar{x}_1 = \bar{x}_2$ or $\lambda_{H_2} = 1$ we cannot accept H_0 if s_1 differs considerably from s_2. If $s_1 = s_2$, $(\lambda_{H_1} = 1)$ then the populations are not the same if $(\bar{x}_1 - \bar{x}_2)$ were large compared to V, which is the estimate based on the sample variance of the standard error of the differences of means.

Thus, the criterion $\lambda_H = \lambda_{H_1}\lambda_{H_2}$ is more crucial than either λ_{H_1} or λ_{H_2} taken separately. Therefore, our conclusion is that λ_H is a reasonable criterion to use for measuring the danger of accepting a false hypothesis.

To control the error of rejecting a true hypothesis, it is necessary to determine the sampling distribution of λ_H. The distributions are derived for λ_{H_1}, and λ_{H_2} and an approximation for λ_H in Neyman and Pearson's paper "On the Problem of Two Samples" (Ref 11:106-109).

The extension of Romanovsky's work to k samples of a single variable was undertaken by Neyman and Pearson's article "On the Problem of k Samples" (Ref 12). The likelihood function for λ_{H_1}, λ_{H_2}, and λ_H were derived by generalizing the two sample deviation. However, methods for calculating the distribution of λ_H and λ_{H_1} were not available at the time of the article. In this case, approximate solutions of the problem were reached by use of the moment coefficients of the λ_H's expressions.

Objective

A further generalization of the problem of k samples of two variables is treated by Pearson and Wilks (Ref 13). Specifically, the problem treated is the case of two correlated variables x and y which have a bivariate normal distribution. The three hypotheses considered are:

1. The hypothesis H_0 that the k populations are identical.

2. The hypothesis H_1 that the samples have come from populations with the same set of variances and correlations, but having means with any differing values whatever.

3. The hypothesis H_2 that the samples are from populations in which the means are equal, when it is assumed that the variances and covariances are equal.

Testing these hypotheses are of great interest to industry; however, the distribution of the test statistics concerning these three hypotheses are not known, and so the problem of finding percentage points of these criteria has thus become difficult. The aim of this thesis is to:

1. Derive the sampling distribution of the test statistic for each of the three hypotheses.

2. Prepare tables of percentage points for $\alpha = .01, .05$ and for $N = 3$ to 100, $k = 2(1)6$.

3. Derive an asymptotic expansion of the distributions which are valid for moderately large values of N.

4. Illustrate the results obtained in this thesis by applying it to actual data.

Chapters II, III, and IV provide preliminary information necessary for the derivation of the sampling distributions of the three hypotheses. In Chapter V the actual derivation of the sampling distribution is undertaken for each hypothesis from which tables of significant levels are obtained. Chapter VI gives an approximation method valid for moderate values of N. Chapter VII uses actual data submitted by the Engineering Division of the Air Force Materiels Laboratory, Wright-Patterson AFB, Ohio, to demonstrate the practical application of the theoretical results.

II. Statistical Preliminaries

Multivariate Normal Distribution

Let the vector $\underline{x}$ have p-components, i.e., $\underline{x} = (x_1, x_2, \ldots, x_p)$, then $\underline{x}$ has a p-variate non-singular normal distribution if its p.d.f. is

$$f(\underline{x}) = \frac{1}{(2\pi)^{\frac{p}{2}} |\underline{\Sigma}|^{\frac{1}{2}}} \exp\left\{\frac{-(\underline{x} - \underline{\mu})' \underline{\Sigma}^{-1} (\underline{x} - \underline{\mu})}{2}\right\} \qquad (2.1)$$

$\underline{\mu}$ and $\underline{\Sigma}$ are the parameters of the distribution; $\underline{\mu}$ is a column vector of elements μ_i $(i = 1,2,\ldots,p)$ and $\underline{\Sigma} = [\sigma_{ij}]$ is a positive definite symmetric matrix of order p. The p.d.f. (2.1) will be denoted by $N_p(\underline{x} \mid \underline{\mu}, \underline{\Sigma})$ and the notation $\underline{x} \sim N_p(\underline{x} \mid \underline{\mu}, \underline{\Sigma})$ will be used to indicate that the variates $\underline{x}$ have a p-variate non-singular normal distribution with parameters $\underline{\mu}$ and $\underline{\Sigma}$. When $\underline{\Sigma}$ is a diagonal matrix, then $f(\underline{x})$ is the product of the p.d.f.s' of p univariate normal variates, showing that the $\underline{x}$'s are independently distributed in that case.

When p = 2, $f(\underline{x})$ has the following bivariate normal p.d.f.

$$f(\underline{x}) = \frac{1}{(2\pi)|\underline{\Sigma}|^{\frac{1}{2}}} \exp\left\{- \frac{tr}{2} (x - \mu)' \underline{\Sigma}^{-1} (x - \mu)\right\}.$$

Let $\underline{x}$ be a random sample of size N from a distribution with p.d.f. $N_p(\underline{x}|\underline{\mu},\underline{\Sigma})$. The vector $\underline{x}$ can be represented as the p by N matrix

$$\underline{x} = \begin{bmatrix} x_{11} x_{12} & \cdots & x_{1N} \\ x_{21} x_{22} & \cdots & x_{2N} \\ \vdots & & \\ x_{p1} x_{p2} & \cdots & x_{pN} \end{bmatrix}$$

The columns of $\underline{x}$ are independently and identically distributed as N_p $(\underline{x}|\underline{\mu},\underline{\Sigma})$. Thus the p.d.f. of $\underline{x}$ is the product of the p.d.f.'s of the N columns of $\underline{x}$.

$$f(x) = \frac{1}{(2\pi)^{\frac{pN}{2}} |\underline{\Sigma}|^{\frac{N}{2}}} \exp\{-\frac{tr}{2} \underline{\Sigma}^{-1}(\underline{x}-\underline{\mu})(\underline{x}-\underline{\mu})'\} \qquad (2.2)$$

where tr is the trace of a matrix.

The exponential term of this p.d.f. is obtained by using the following property concerning the trace of the product of matrices.

Let P, Q, and R be matrices such that the product PQR exist. Then

$$tr(PQR) = tr(RPQ) = tr(QRP) = tr(QRP)$$

Specifically, since $(\underline{x} - \underline{\mu})'$, $\underline{\Sigma}^{-1}$, $(\underline{x} - \underline{\mu})$ are matrices whose product exists,

$$\mathrm{tr}(\underline{x} - \underline{\mu})' \underline{\Sigma}^{-1}(\underline{x} - \underline{\mu}) = \mathrm{tr}(\underline{x} - \underline{\mu})(\underline{x} - \underline{\mu})'\underline{\Sigma}^{-1}$$

$$= \mathrm{tr}\underline{\Sigma}^{-1}(\underline{x} - \underline{\mu})(\underline{x} - \underline{\mu})' \quad (2.3)$$

Maximum Likelihood Estimates (MLE) of $\underline{\mu}$ and $\underline{\Sigma}$

Let $\underline{x}$ be a random sample of N observations, where $\underline{x} \sim N_p(\underline{x}|\underline{\mu},\underline{\Sigma})$, $N > p$. The likelihood function of $\underline{x}$ is

$$L = \frac{1}{(2\pi)^{\frac{pN}{2}} |\underline{\Sigma}|^{\frac{N}{2}}} \exp\{-\frac{1}{2} \sum_{\alpha=1}^{N} (\underline{x}_\alpha - \underline{\mu})' \underline{\Sigma}^{-1}(\underline{x}_\alpha - \underline{\mu})\} \quad (2.4)$$

To find the MLE of $\underline{\mu}$ and $\underline{\Sigma}$ it is necessary to maximize the likelihood function L. Since the likelihood function L and its logarithm are maximized for the same value we will consider log L.

$$\log L = \frac{-1}{2}pN \log(2\pi) + \frac{1}{2}N \log\underline{\Sigma}^{-1}$$

$$- \frac{1}{2} \sum_{\alpha=1}^{N} (\underline{x}_\alpha - \underline{\mu})' \underline{\Sigma}^{-1}(\underline{x}_\alpha - \underline{\mu}) \quad (2.5)$$

The following properties will enable us to rewrite equation (2.5) in a form which is easily maximized.

<u>Definition 1</u>: Let the sample mean be defined as:

$$\underline{\bar{x}} = \frac{1}{N} \sum_{\alpha=1}^{N} \underline{x}_\alpha = \begin{bmatrix} \frac{1}{N} \sum_{\alpha=1}^{N} x_{1\alpha} \\ \vdots \\ \frac{1}{N} \sum_{\alpha=1}^{N} x_{p\alpha} \end{bmatrix} = \begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{X}_p \end{bmatrix} \tag{2.6}$$

Definition 2: The matrix of sums of squares and cross products of deviations about the mean is defined as $\underline{A}$ where

$$\underline{A} = \sum_{\alpha=1}^{N} (\underline{x}_\alpha - \underline{\bar{x}})(\underline{x}_\alpha - \underline{\bar{x}})' \tag{2.7}$$

Lemma 2.1: Let $\underline{x}_1, \underline{x}_2, \ldots \underline{x}_N$ be N p-component vectors and let $\underline{\bar{x}}$ be defined by definition 1. *Then for any* vector $\underline{b}$

$$\sum_{\alpha=1}^{N} (\underline{x}_\alpha - \underline{b})(\underline{x}_\alpha - b)' = \sum_{\alpha=1}^{N} (\underline{x}_\alpha - \underline{\bar{x}})(\underline{x}_\alpha - \underline{\bar{x}})' + N(\underline{\bar{x}} - \underline{b})(\bar{x} - \underline{b})' \tag{2.8}$$

proof: (Ref 1:46)

Lemma 2.2: Let $f(\underline{C}) = N \log |\underline{C}| - \mathrm{tr}\, \underline{CD}$ where $\underline{C} = (c_{ij})$ and $\underline{D} = (d_{ij})$ are both positive semi-definite. Then the maximum of $f(\underline{C})$ is taken at

$$\underline{C} = N\, \underline{D}^{-1}$$

proof: (Ref 1:47)

Now, by using the property tr(a) = a where a is a scaler and applying equation (2.3), we can rewrite equation (2.5) in the following form:

$$\sum_{\alpha=1}^{N} (\underline{x}_\alpha - \underline{\mu})' \underline{\Sigma}^{-1} (\underline{x}_\alpha - \underline{\mu}) = \text{tr} \sum_{\alpha=1}^{N} \underline{\Sigma}^{-1} (\underline{x}_\alpha - \underline{\mu})(\underline{x}_\alpha - \mu)' \tag{2.9}$$

By using lemma 2.1 and setting $\underline{b} = \underline{\mu}$ (2.8) becomes

$$= \text{tr}\ \underline{\Sigma}^{-1}\underline{A} + \text{tr}N(\underline{\bar{x}} - \underline{\mu})' \underline{\Sigma}^{-1}(\underline{\bar{x}} - \underline{\mu})$$

Thus

$$\sum_{\alpha=1}^{N} (\underline{x}_\alpha - \underline{\mu})' \underline{\Sigma}^{-1} (\underline{x}_\alpha - \underline{\mu}) = \text{tr}\ \underline{\Sigma}^{-1}\ \underline{A}$$

$$+ \text{tr}\ N(\underline{\bar{x}}_\alpha - \underline{\mu})' \underline{\Sigma}^{-1} (\underline{\bar{x}} - \underline{\mu}) \tag{2.10}$$

Now substituting the RHS of (2.10) in equation (2.5) gives us a form that is easy to maximize.

$$\log L = \tfrac{1}{2}pN \log(2\pi) + \tfrac{1}{2}\log \underline{\Sigma}^{-1} - \tfrac{1}{2}\text{tr}\ \underline{\Sigma}^{-1}\ \underline{A}$$

$$- \tfrac{1}{2}N\ (\underline{\bar{x}} - \underline{\mu})' \underline{\Sigma}^{-1}(\underline{\bar{x}} - \underline{\mu}) \tag{2.11}$$

The first term of (2.11) is a constant and is therefore already at its maximum value. The last term is at its maximum value of zero when $\underline{\mu} = \underline{\bar{x}}$. Since the remaining terms are not functions of $\underline{\mu}$, the MLE of $\underline{\mu}$ denoted $\hat{\underline{p}}$ is $\underline{\bar{x}}$.

To find the MLE for $\underline{\Sigma}^{-1}$ notice that the second and third term of (2.11) are functions of $\underline{\Sigma}^{-1}$ alone and

therefore can be maximized by applying lemma 2.2, putting $\underline{\Sigma}^{-1}$ for $\underline{C}$ and $\underline{A}$ for $\underline{D}$. Thus, the maximum of log L occurs when

$$\underline{\Sigma}^{-1} = \frac{1}{N}\,\underline{A}^{-1}$$

To summarize, the MLE for $\underline{\mu}$ and $\underline{\Sigma}$ are

$$\hat{\underline{\mu}} = \underline{\bar{x}} \qquad \hat{\underline{\Sigma}} = \frac{\underline{A}}{N} \tag{2.12}$$

The Wishart Distribution

Let $\underline{x} = (\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_N)$ be a random sample of size N from $N_p\ (\underline{x}\,|\underline{0}, \underline{\Sigma})$. The Wishart matrix $\underline{A}$ is defined as the p x p symmetric matrix of sums of squares and sums of products of the sample observations.

Let

$$\underline{A} = \sum_{\alpha=1}^{N} (\underline{x}_\alpha - \underline{\bar{x}})(\underline{x}_\alpha - \underline{\bar{x}})'$$

Then it is known that $\underline{A}$ has the following p.d.f. (known as the Wishart Distribution with n = N - 1 degrees of freedom). (Ref 1:54)

$$f(\underline{A}) = K(\underline{\Sigma}, n)|\underline{A}|^{\left(\frac{n}{2} - \frac{p+1}{2}\right)} \exp\left(\frac{-1}{2}\mathrm{tr}\ \underline{\Sigma}^{-1}\ \underline{A}\right) \tag{2.13}$$

where $\underline{A} > 0$ (i.e., A is positive definite) and

$$\Gamma_p\left(\frac{n}{2}\right) = \pi^{\frac{p(p-1)}{4}} \prod_{i=1}^{p} \Gamma\left(\frac{(n+1-i)}{2}\right)$$

$$K(\underline{\Sigma},n) = \frac{1}{2^{\frac{np}{2}}\Gamma_p\left(\frac{n}{2}\right)|\underline{\Sigma}|^{\frac{n}{2}}}$$

The distribution of $\underline{A}$ is $W(\underline{A}|\underline{\Sigma},n)$ where n represents the degrees of freedom. note: When p = 1, the Wishart distribution is a Chi-Square with n degrees of freedom.

Theorems Concerning the Wishart Distribution

The following theorems are necessary in order to derive the results in later chapters.

Theorem 2.1: If $\underline{A}_1,\underline{A}_2,\ldots,\underline{A}_g$ are matrices, each independently distributed as $W(\underline{A}_g|\underline{\Sigma},n_g)$ then

$$\underline{A} = \sum_{i=1}^{g} \underline{A}_i$$

is distributed $W(\underline{A}|\underline{\Sigma}, \sum_{i=1}^{g} n_i)$

proof: (Ref 1:162)

Theorem 2.2: Let $\underline{A}$ and $\underline{T}$ both be p by p positive matrices, then

$$\int_{\underline{A}>0} \underline{A}^{\left(\frac{\alpha}{2} - \frac{(p+1)}{2}\right)} \exp\left(\frac{-1}{2} \operatorname{tr} \underline{T}^{-1} \underline{A}\right) dA = K^{-1}(\underline{T},\alpha)$$

where

$$K(\underline{T},\alpha) = \frac{T^{-\frac{\alpha}{2}}}{2^{p\frac{\alpha}{2}}\Gamma_p\left(\frac{\alpha}{2}\right)}$$

and

$$\Gamma_p\left(\frac{\alpha}{2}\right) = \pi^{\frac{p(p-1)}{4}} \prod_{i=1}^{p} \Gamma\left(\frac{(\alpha+1-i)}{2}\right)$$

This result follows directly from the Wishart p.d.f. since multiplying both sides by $K(\underline{T},\alpha)$ gives the Wishart p.d.f. Thus, if we denote the function under the integral as $f(\underline{A})$ we have

$$K(\underline{T},\alpha) \int_{\underline{A}>0} f(\underline{A})\, dA = 1$$

Theorem 2.3: Let $\underline{A} \sim W(\underline{A}|\underline{\Sigma},n)$ then

$$E(|\underline{A}|^h) = \frac{K(\underline{\Sigma},n)}{K(\underline{\Sigma},n+2h)}$$

where $K(\underline{\Sigma},n)$ is defined as in theorem 2.2.

Proof:

$$E(\underline{A}^h) = \int_{\underline{A}>0} |\underline{A}|^h f(\underline{A})\, dA = \int_{A>0} |\underline{A}|^h K(\underline{\Sigma},n)$$

$$|A|^{\left(\frac{n}{2} - \frac{p+1}{2}\right)} \exp\left(-\frac{1}{2}\mathrm{tr}\ \underline{\Sigma}^{-1}\underline{A}\right) dA$$

$$= K(\underline{\Sigma},n) \int_{\underline{A}>0} |\underline{A}|^{\left(h + \frac{n}{2} - \frac{p+1}{2}\right)} \exp\left(-\frac{1}{2}\mathrm{tr}\ \underline{\Sigma}^{-1}\underline{A}\right) dA$$

$$= K(\underline{\Sigma},n) \int_{\underline{A}>0} |\underline{A}|^{\{\frac{1}{2}(n + h) - \frac{p+1}{2}\}} \exp\left(-\frac{1}{2}\mathrm{tr}\,\underline{\Sigma}^{-1}\underline{A}\right) dA$$

$$= K(\underline{\Sigma},n)K^{-1}(\underline{\Sigma},n + 2h) = \frac{K(\underline{\Sigma},n)}{K(\underline{\Sigma}, n + 2h)}$$

Q.E.D.

Theorem 2.4: Let $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_N$ $(N > p + 1)$ be independently distributed as $N_p(x|\underline{\mu},\underline{\Sigma})$. Then the distribution of

$$\underline{A} = \sum_{\alpha=1}^{N} (\underline{x}_\alpha - \bar{\underline{x}})(\underline{x}_\alpha - \bar{\underline{x}})'$$

is $W(\underline{A}|\underline{\Sigma},n)$ where $n = N - 1$.

Proof: (Ref 1:59)

Maximum Likelihood Ratio Test

To test a composite hypothesis against an alternative hypothesis, the likelihood ratio test is used. Let $\underline{x}$ be independently distributed as $N_p(\underline{x}|\underline{\mu},\underline{\Sigma})$ and the parameter space $\Omega = (\underline{\mu},\underline{\Sigma})$. Let ω be a subset of Ω restricted under any null hypothesis, i.e. $\omega = (\tilde{\underline{\mu}},\tilde{\underline{\Sigma}})$. From the method of MLE equation (2.4), if $\underline{x} \sim N_p(\underline{x}|\underline{\mu},\underline{\Sigma})$ the likelihood function for $\underline{\mu}$ and $\underline{\Sigma} \in \Omega$ is

$$L(\Omega) = \frac{1}{(2\pi)^{\frac{pN}{2}} |\underline{\Sigma}|^{\frac{N}{2}}} \exp\left(-\frac{1}{2}\sum_{\alpha=1}^{N}(\underline{x}_\alpha - \underline{\mu})'\,\underline{\Sigma}^{-1}(\underline{x}_\alpha - \underline{\mu})\right) \quad (2.14)$$

Suppose $\omega \subset \Omega$ where $\underline{\tilde{\mu}}, \underline{\tilde{\Sigma}} \in \omega$, then the likelihood function for ω is

$$L(\omega) = \frac{1}{(2\pi)^{\frac{pN}{2}} |\underline{\tilde{\Sigma}}|^{\frac{N}{2}}} \exp\{-\frac{1}{2} \sum_{\alpha=1}^{N} (\underline{x}_\alpha - \underline{\tilde{\mu}})' \underline{\tilde{\Sigma}}^{-1} (\underline{x}_\alpha - \underline{\tilde{\mu}})\} \tag{2.15}$$

Denote the maximum of $L(\Omega)$ in Ω by $L(\hat{\Omega})$ and denote the maximum $L(\omega)$ in ω by $L(\hat{\omega})$. Then the likelihood ratio criteria is

$$\lambda = \frac{\max_\omega L(\omega)}{\max_\Omega L(\Omega)} = \frac{L(\hat{\omega})}{L(\hat{\Omega})} \tag{2.16}$$

The statistic λ will be our criteria for hypotheses testing. As explained earlier the value of λ is between 0 and 1, and a small ratio of λ leads to rejecting our hypothesis where a ratio near unity gives strong support for not rejecting.

The subject of the next chapter is to determine the criterion λ for the three separate hypotheses.

III. Derivation of the Criteria

Let $\underline{x}_g (g = 1,2,\ldots,k)$ be each independent and identically distributed as $N_p(\underline{x}_g | \underline{\mu}^g, \underline{\Sigma}_g)$. In this chapter we shall derive the likelihood ratio criteria for the following hypotheses:

$$\begin{aligned}
&\text{[i]} \quad H_0: \underline{\mu}^1 = \underline{\mu}^2, = \ldots = \underline{\mu}^k \\
&\qquad\quad \underline{\Sigma}_1 = \underline{\Sigma}_2, = \ldots = \underline{\Sigma}_k \\
&\text{[ii]} \quad H_1: \underline{\Sigma}_1 = \underline{\Sigma}_2, = \ldots = \underline{\Sigma}_k \\
&\text{[iii]} \quad H_2: \underline{\mu}^1 = \underline{\mu}^2, = \ldots = \underline{\mu}^k \text{ given } \underline{\Sigma}_1 = \underline{\Sigma}_2, = \ldots \underline{\Sigma}_k
\end{aligned} \tag{3.1}$$

Let $\underline{x}^g$ be a random sample of size N_g from the g^{th} population, then using definition (2.6), the mean $\overline{\underline{x}}^g$ of the g^{th} population is denoted by

$$\overline{\underline{x}}^g = \frac{1}{N_g} \sum_{\alpha=1}^{N} x_\alpha^g$$

Let the combined mean $\overline{\underline{x}}$ of all the sample populations be denoted by

$$\overline{\underline{x}} = \sum_{g=1}^{k} \frac{N_g \overline{\underline{x}}^g}{\sum_{g=1}^{k} N_g} = \sum_{g=1}^{k} \frac{N_g \overline{\underline{x}}^g}{N}$$

where

$$N = \sum_{g=1}^{k} N_g$$

Let $\underline{A}_g$ denote the matrix of sums of squares and products for the g^{th} population, as defined in (2.7); i.e.,

$$\underline{A}_g = \sum_{\alpha=1}^{N_g} (\underline{x}_\alpha^g - \overline{\underline{x}}^g)(\underline{x}_\alpha^g - \overline{\underline{x}}^g)' \quad g = 1,2,\ldots,k \quad (3.2)$$

<u>Likelihood Estimates of</u>
<u>$\underline{\mu}^g$ and $\underline{\Sigma}_g$</u>

The k populations are independent; therefore, the likelihood function of all the sample observations is the product of the separate likelihood functions; so generalizing the results in Chapter II to k populations the MLE for $\underline{\mu}^g$ and $\underline{\Sigma}_g$ become

$$\hat{\underline{\mu}}^g = \overline{\underline{x}}^g \qquad \hat{\underline{\Sigma}}_g = \frac{\underline{A}_g}{N_g}$$

The likelihood function for our parameter space $\Omega = (\underline{\mu}^1, \underline{\mu}^2, \ldots, \underline{\mu}^k, \underline{\Sigma}_1, \underline{\Sigma}_2, \ldots, \underline{\Sigma}_k)$, where $(-\infty < \underline{\mu}^i < \infty)$, and $(0 < \underline{\Sigma}_i < \infty)$, $i = 1,2,\ldots,k$ is

$$L(\Omega) = \frac{1}{\prod_{g=1}^{k} (2\pi)^{\frac{pN_g}{2}} |\underline{\Sigma}|^{\frac{N_g}{2}}} \exp \{-\frac{1}{2} \sum_{g=1}^{k} \operatorname{tr} \underline{\Sigma}_g^{-1} \underline{A}_g$$

$$- \frac{1}{2} \sum_{g=1}^{k} N_g (\overline{\underline{x}}^g - \underline{\mu}^g)' \underline{\Sigma}_g^{-1} (\overline{\underline{x}}^g - \underline{\mu}^g)\}.$$

Taking the logarithm of $L(\Omega)$ gives

$$\log L(\Omega) = \sum_{g=1}^{k} \left(-\frac{pN}{2}\log(2\pi) + \sum_{g=1}^{k} \frac{N_g}{2} \log|\underline{\Sigma}_g^{-1}|\right.$$

$$- \frac{1}{2} \sum_{g=1}^{k} \mathrm{tr}\, \underline{\Sigma}_g^{-1} \underline{A}_g$$

$$- \frac{1}{2} \mathrm{tr} \sum_{g=1}^{k} N_g (\bar{\underline{x}}^g - \underline{\mu}^g)' \underline{\Sigma}_g^{-1} (\bar{\underline{x}}^g - \underline{\mu}^g)$$

Setting $\sum_{g=1}^{k} N_g = N$ and bringing the exponent of $\underline{\Sigma}_g^{-1}$ in front of the summation gives

$$\log L(\Omega) = -\frac{pN}{2} \log(2\pi) - \frac{1}{2} \sum_{g=1}^{k} N_g |\underline{\Sigma}_g|$$

$$-\frac{1}{2} \sum_{g=1}^{k} \mathrm{tr}\, \underline{\Sigma}_g^{-1} \underline{A}_g$$

$$-\frac{1}{2} \mathrm{tr} \sum_{g=1}^{k} N_g (\bar{\underline{x}}^g - \underline{\mu}^g)' \underline{\Sigma}_g^{-1} (\bar{\underline{x}}^g - \underline{\mu}^g) \qquad (3.3)$$

The last term on the RHS has its maximum value when $\underline{\mu}^g = \bar{\underline{x}}^g$ and thus substituting this, brings the term to zero. Also, substituting the MLE of $\underline{\Sigma}_g$ in equation (3.3) gives the maximum value of $L(\Omega)$. Thus

$$\log L(\hat{\Omega}) = -\frac{pN}{2}\log(2\pi) - \frac{1}{2}\sum_{g=1}^{k} N_g \log\left|\frac{A_g}{N_g}\right|$$

$$-\frac{1}{2}\sum_{g=1}^{k} \mathrm{tr}(N_g A_g^{-1}) A_g$$

The last term becomes $\frac{-1}{2}\sum_{g=1}^{k} N_g$ tr I. Since tr I = p and $\sum_{g=1}^{k} N_g = N$ we have

$$\log L(\hat{\Omega}) = \log(2\pi)^{-\frac{pN}{2}} + \sum_{g=1}^{k} \log\left|\frac{A_g}{N_g}\right|^{-\frac{N_g}{2}}$$

$$+ \log\{\exp(-\frac{pN}{2})\}$$

$$= \log(2\pi)^{-\frac{pN}{2}} \log\left(\prod_{g=1}^{k} \left|\frac{A_g}{N_g}\right|^{\frac{N_g}{2}}\right)$$

$$+ \log\{\exp(-\frac{pN}{2})\}$$

$$= \log\left[(2\pi)^{-\frac{pN}{2}} \prod_{g=1}^{k} \left|\frac{A_g}{N_g}\right|^{-\frac{N_g}{2}} \exp(-\frac{pN}{2})\right]$$

$$L(\hat{\Omega}) = (2\pi)^{-\frac{pN}{2}} \prod_{g=1}^{k} \left|\frac{A_g}{N_g}\right|^{-\frac{N_g}{2}} \exp(-\frac{pN}{2}) \qquad (3.4)$$

Derivation of Criteria for H_0

To test the hypothesis

$$H_0: \underline{\mu}^1 = \underline{\mu}^2 = \dots = \underline{\mu}^k$$
$$\underline{\Sigma}_1 = \underline{\Sigma}_2 = \dots = \underline{\Sigma}_k$$

substitute parameters $\underline{\mu}^g = \underline{\mu}$ and $\underline{\Sigma}_g = \underline{\Sigma}$ into the likelihood function. Thus, equation (3.3) becomes

$$\log L(\omega_0) = -\frac{pN}{2}\log(2\pi) - \frac{1}{2}\sum_{g=1}^{k} N_g \log|\underline{\Sigma}| - \frac{1}{2}\sum_{g=1}^{k} \mathrm{tr}\,\underline{\Sigma}^{-1}\underline{A}_g$$

$$-\frac{1}{2}\sum_{g=1}^{k} \mathrm{tr}\,\underline{\Sigma}^{-1} N_g(\underline{\bar{x}}^g - \underline{\mu})(\underline{\bar{x}}^g - \underline{\mu})'$$

Now, substituting $\sum_{g=1}^{k} N_g = N$ and $\sum_{g=1}^{k} \underline{A}_g = \underline{A}$ gives

$$\log L(\omega_0) = -\frac{pN}{2}\log(2\pi) - \frac{N}{2}\log|\underline{\Sigma}| - \frac{1}{2}\mathrm{tr}\,\underline{\Sigma}^{-1}\underline{A}$$

$$-\frac{1}{2}\mathrm{tr}\,\underline{\Sigma}^{-1}\left[\sum_{g=1}^{k} N_g(\underline{\bar{x}}^g - \underline{\mu})(\underline{\bar{x}}^g - \underline{\mu})'\right] \qquad (3.5)$$

Combining the last two terms and applying lemma 2.1 with $\underline{b} = \underline{\mu}$, equation (3.5), can be written as

$$\log L(\omega_0) = -\frac{pN}{2}\log(2\pi) - \frac{N}{2}\log|\underline{\Sigma}| - \frac{1}{2}\mathrm{tr}\,\underline{\Sigma}^{-1}$$

$$\left[\underline{A} + \sum_{g=1}^{k} N_g(\underline{\bar{x}}^g - \underline{b})(\underline{\bar{x}}^g - \underline{b})'\right] \qquad (3.6)$$

The term in the bracket becomes

$$\underline{A} + \sum_{g=1}^{k} N_g(\underline{\bar{x}}^g - \underline{\bar{x}})(\underline{\bar{x}}^g - \underline{\bar{x}})' + N(\underline{\bar{x}} - \underline{\mu})(\underline{\bar{x}} - \underline{\mu})' \qquad (3.7)$$

Thus, the $\log L(\omega_0)$ is maximized with $\hat{\underline{\mu}} = \underline{\bar{x}}$.

For convenience we will denote $\sum_{g=1}^{k} N_g(\underline{\bar{x}}^g - \underline{\bar{x}})(\underline{\bar{x}}^g - \underline{\bar{x}})'$ as B.

To find the estimate for $\underline{\Sigma}$ it is necessary to maximize equation (3.5) with respect to $\underline{\Sigma}^{-1}$.

$$f(\Sigma^{-1}) = -\frac{pN}{2}\log(2\pi) + \frac{N}{2}\log|\underline{\Sigma}^{-1}| - \frac{1}{2}\operatorname{tr}\underline{\Sigma}^{-1}(\underline{A} + \underline{B}) \tag{3.8}$$

Applying lemma 2.2 on the last two terms with $\underline{C} = \underline{\Sigma}^{-1}$ and $\underline{A} + \underline{B} = \underline{D}$ the maximum of $f(\underline{\Sigma}^{-1})$ is at $\underline{C} = N\,\underline{D}^{-1}$ or $\underline{\Sigma}^{-1} = N(\underline{A}+\underline{B})^{-1}$. Thus, the estimate for $\underline{\Sigma}$ is

$$\hat{\underline{\Sigma}} = \frac{\underline{A} + \underline{B}}{N}$$

Now, substituting $\hat{\underline{\Sigma}}$ for $\underline{\Sigma}$ in equation (3.8) gives

$$\begin{aligned} L(\hat{\omega}_0) &= -\frac{pN}{2}\log(2\pi) - \frac{N}{2}\log\left|\frac{\underline{A} + \underline{B}}{N}\right| \\ &\quad -\frac{1}{2}\operatorname{tr}\left(\frac{\underline{A} + \underline{B}}{N}\right)^{-1}(\underline{A} + \underline{B}) \\ &= (2\pi)^{-\frac{pN}{2}}\left|\frac{\underline{A} + \underline{B}}{N}\right|^{-\frac{N}{2}}\exp\left(-\frac{pN}{2}\right) \end{aligned} \tag{3.9}$$

Therefore, the criteria λ_0, for testing hypothesis H_0 is the likelihood ratio

$$\lambda_0 = \frac{L(\hat{\omega}_0)}{L(\hat{\Omega})} = \prod_{g=1}^{k} \frac{|\underline{A}_g|^{\frac{N_g}{2}} N^{\frac{pN}{2}}}{|\underline{A}+\underline{B}|^{\frac{N}{2}} \prod_{g=1}^{k} N_g^{\frac{pN_g}{2}}} \tag{3.10}$$

Derivation of Criteria for H_1

To test the hypothesis

$H_1: \underline{\Sigma}_1 = \underline{\Sigma}_2 = \ldots = \underline{\Sigma}_k$ the means any value our parameter space ω_1 belongs to Ω, where $(0 < \underline{\Sigma}_1 = \underline{\Sigma}_2 = \ldots = \underline{\Sigma}_k < \infty)$. Thus we substitute $\underline{\Sigma}$ for $\underline{\Sigma}_g$ in equation (3.3) to get

$$\log L(\omega_1) = -\frac{pN}{2}\log(2\pi) - \frac{1}{2}\sum_{g=1}^{k} N_g \log|\underline{\Sigma}| - \frac{1}{2}\sum_{g=1}^{k} \operatorname{tr} \underline{\Sigma}^{-1} \underline{A}_g$$

$$-\frac{1}{2}\operatorname{tr}\sum_{g=1}^{k} N_g (\underline{\bar{x}}^g - \underline{\mu})^g)' \underline{\Sigma}_g^{-1} (\underline{\bar{x}}^g - \underline{\mu}^g) \tag{3.11}$$

The last term is maximized when $\underline{\mu}^g = \underline{\bar{x}}^g$. By substituting $\sum_{g=1}^{k} N_g = N$ and $\underline{A} = \sum_{g=1}^{k} \underline{A}_g$ and bringing the trace in front of the summation we have

$$\log L(\omega_1) = -\frac{pN}{2}\log(2\pi) - \frac{N}{2}\log|\underline{\Sigma}| - \frac{1}{2}\operatorname{tr} \underline{\Sigma}^{-1} \underline{A}$$

Now using lemma 2.2, we have $\underline{\Sigma}^{-1} = N\underline{A}^{-1}$ or $\underline{\Sigma} = \frac{\underline{A}}{N}$. Using $\frac{\underline{A}}{N}$ for $\underline{\Sigma}$ the maximized function becomes

$$\log L(\hat{\omega}_1) = -\frac{pN}{2}\log(2\pi) - \frac{N}{2}\log\left|\frac{\underline{A}}{N}\right| - \frac{1}{2}\operatorname{tr}\left(\frac{\underline{A}}{N}\right)^{-1} \underline{A}$$

$$= -\frac{pN}{2}\log(2\pi) - \frac{N}{2}\log\left|\frac{\underline{A}}{N}\right| - \frac{1}{2}\operatorname{tr} NI$$

Using tr I = p and rewritting gives

$$\log L(\hat{\omega}) = \log(2\pi)^{-\frac{pN}{2}} + \log\left|\frac{\underline{A}}{N}\right|^{-\frac{N}{2}} \exp(-\frac{pN}{2})$$

Therefore, using exponentiation gives

$$L(\hat{\omega}_1) = (2\pi)^{-\frac{pN}{2}}\left|\frac{\underline{A}}{N}\right|^{-\frac{N}{2}} \exp(-\frac{pN}{2}) \tag{3.12}$$

The likelihood ratio criteria λ_1 for testing the hypothesis H_1 is given by

$$\lambda_1 = \frac{L(\hat{\omega}_1)}{L(\hat{\Omega})} = \prod_{g=1}^{k} \frac{|\underline{A}_g|^{\frac{N_g}{2}} N^{\frac{pN}{2}}}{|\underline{A}|^{\frac{N}{2}} \prod_{g=1}^{k} N_g^{\frac{pN_g}{2}}} \tag{3.13}$$

Derivation of Criteria for H_2

To test hypothesis

H_2: $\underline{\mu}^1 = \underline{\mu}^2 = \ldots \underline{\mu}^k$ given that $\underline{\Sigma}_1 = \underline{\Sigma}_2 \ldots = \underline{\Sigma}_k$ note that the log of the likelihood function equation (3.3) with the condition $\underline{\Sigma}_1 = \underline{\Sigma}_2 = \ldots = \underline{\Sigma}_k$ imposed is the same as log $L(\omega_1)$, equation (3.12). Further, when the restriction $\underline{\mu}^1 = \underline{\mu}^2 = \ldots \underline{\mu}^k$ is imposed the log $L(\omega_2)$ is the same as log $L(\omega_0)$ given in equation (3.9). Thus, the criteria λ_2, can be represented as the ratio of these two equations:

$$\lambda_2 = \frac{L(\hat{\omega}_0)}{L(\hat{\omega}_1)} = \frac{|\underline{A}|^{\frac{N}{2}}}{|\underline{A}+\underline{B}|^{\frac{N}{2}}} = \left(\frac{|\underline{A}|}{|\underline{A}+\underline{B}|}\right)^{\frac{N}{2}} \tag{3.14}$$

Particular Cases

In this thesis we will be concerned with equal sample sizes and $p = 2$. Thus, when $N_1 = N_2 = \ldots N_k = n$ and $N = kn$ the three criteria take the form:

$$\lambda_0 = k^{kn} \prod_{g=1}^{k} \left[\frac{|\underline{A}_g|}{|\underline{A}+\underline{B}|} \right]^{\frac{n}{2}} \tag{3.15}$$

$$\lambda_1 = k^{kn} \prod_{g=1}^{k} \left[\frac{|\underline{A}_g|}{|\underline{A}|} \right]^{\frac{n}{2}} \tag{3.16}$$

$$\lambda_2 = \left[\frac{|\underline{A}|}{|\underline{A}+\underline{B}|} \right]^{\frac{kn}{2}} \tag{3.17}$$

In this chapter we derived the likelihood ratio criteria to test each of the three hypotheses. Next, we must determine a critical region for testing H_i $i = 0,1,2$. Our critical region is the set defined by $(0 < \lambda < \phi)$ and our decision rule is to reject H_i if $\lambda < \phi$. The function λ defines a random variable $\lambda(\underline{x}_1, \underline{x}_2, \ldots, \underline{\underline{x}}_N)$ and the significance level of the test is given by

$$\alpha = \Pr[\lambda < \phi;\ H_i]$$

We determine these probabilities by finding the sampling distribution of our likelihood ratio criteria λ_0, λ_1, and λ_2.

In the next chapter we will obtain the moments of the criteria from which the sampling distributions will be derived in Chapter V.

IV. Derivation of the Moments

To obtain the sampling distributions of the criteria λ_0, λ_1 and λ_2, we need to obtain their h^{th} moments which are derived below.

The h^{th} Moment of λ_0

The h^{th} moment of λ_0 is

$$E(\lambda_0^h) = K_0 E(L_0^h)$$

where
$$K_0 = \frac{N^{\frac{kNh}{2}}}{\prod_{g=1}^{k} N_g^{\frac{N_g h}{2}}} \quad \text{and} \quad L_0 = \prod_{g=1}^{k} \frac{|\underline{A}_g|^{\frac{N_g}{2}}}{|\underline{A}+\underline{B}|^{\frac{N}{2}}} \tag{4.1}$$

By theorem 2.4, $\underline{A}_g \sim W(\underline{A}_g|\underline{\Sigma},n_g)$. The matrix $\underline{B}$ is the sum of squares and sum of products between means for k samples, thus $\underline{B} \sim W(\underline{B}|\underline{\Sigma},k-1)$. Theorem 3.3.2 (Ref 1:53) establishes the independence of the sample means and covariance matrices and since $\underline{B} = \sum_{g=1}^{k} N_g(\underline{\bar{x}}^g - \underline{\bar{x}})(\underline{\bar{x}}^g - \underline{\bar{x}})'$ it follows that $\underline{A}$ and $\underline{B}$ are independent.

Since L_0 is a function of $\underline{A}_g$ and $\underline{B}$ we have by definition

$$E(L_0^h) = \int_{\underline{A}_g>0,\underline{B}>0} \prod_{g=1}^{k} \frac{|\underline{A}_g|^{\frac{N_g h}{2}}}{|\underline{A}+\underline{B}|^{\frac{Nh}{2}}} \prod_{g=1}^{k} W(\underline{A}_g|\underline{\Sigma},n_g)W(\underline{B}|\underline{\Sigma},k-1)\,dA_g\,dB$$

$$= \prod_{g=1}^{k} K(\underline{\Sigma},n_g) \int_{\underline{A}_g>0,\underline{B}>0} |\underline{A}+\underline{B}|^{-\frac{Nh}{2}} \prod_{g=1}^{k} |\underline{A}_g|^{\frac{N_g h+n_g}{2} - (\frac{p+1}{2})}$$

$$\exp(-\frac{1}{2}\,\mathrm{tr}\,\underline{\Sigma}^{-1}\underline{A}_g)W(\underline{B}|\underline{\Sigma},k-1)\,dA_g\,dB$$

$$= \prod_{g=1}^{k} \frac{K(\underline{\Sigma},n_g)}{K(\underline{\Sigma},N_g h+n_g)} \int_{\underline{A}_g>0,\underline{B}>0} |\underline{A}+\underline{B}|^{-\frac{Nh}{2}} \prod_{g=1}^{k} W(\underline{A}_g|\underline{\Sigma},N_g h+n_g)$$

$$W(\underline{B}|\underline{\Sigma},k-1)\,dA_g\,dB \tag{4.2}$$

The integral on the right hand side of (4.2) is equal to $E(|\underline{A}+\underline{B}|^{-\frac{hN}{2}})$ since $\underline{A} = \underline{A}_1 + \underline{A}_2 + \ldots + \underline{A}_k$ and $\underline{B}$ are independent. Therefore,

$$E(L_0^h) = \prod_{g=1}^{k} \frac{K(\underline{\Sigma},n_g)}{K(\underline{\Sigma},N_g h+n_g)} \cdot E(|\underline{A}+\underline{B}|)^{-\frac{Nh}{2}} \tag{4.3}$$

Recall that $\underline{A} \sim W\{\underline{A}|\underline{\Sigma}, \sum_{g=1}^{k} (N_g h + n_g)\}$ and $\underline{B} \sim W(\underline{B}|\underline{\Sigma},k-1)$,

so by theorem 2.1 $(\underline{A}+\underline{B}) \sim W\{(\underline{A}+\underline{B})|\underline{\Sigma}, \sum_{g=1}^{k}(N_g h + n_g) + k-1\}$.

Now $n_g = N_g - 1$ and $\sum_{g=1}^{k} N_g = N$. Therefore,

$$\sum_{g=1}^{k}(N_g h + n_g) + k - 1 = Nh + N - k + k - 1 = Nh + N - 1.$$

Hence, $(\underline{A}+\underline{B}) \sim W\{(\underline{A}+\underline{B})|\underline{\Sigma}, Nh + N - 1\}$. Thus, using theorem 2.3, substituting $(\underline{A}+\underline{B})$ for $\underline{A}$ and $-\frac{Nh}{2}$ for h we have

$$E(|\underline{A}+\underline{B}|^{-\frac{Nh}{2}}) = \frac{K(\underline{\Sigma}, Nh + N - 1)}{K(\underline{\Sigma}, Nh + N - 1 + 2(-\frac{Nh}{2}))} = \frac{K(\underline{\Sigma}, Nn + N - 1)}{K(\underline{\Sigma}, N - 1)} \tag{4.4}$$

From equations (4.3) and (4.4) we have

$$E(\lambda_0^h) = K_0 \prod_{g=1}^{k} \frac{K(\underline{\Sigma}, n_g)}{K(\underline{\Sigma}, N_g h + n_g)} \cdot \frac{K(\underline{\Sigma}, Nh + N - 1)}{K(\underline{\Sigma}, N - 1)} \tag{4.5}$$

Using the definition $\Gamma_p(\frac{n}{2})$ and $K(\underline{\Sigma}, n)$ from equation (2.13) we have

$$E(\lambda_0^h) = \frac{N^{\frac{kNh}{2}}}{\prod_{g=1}^{k} N_g^{\frac{kN_g h}{2}}} \prod_{i=1}^{p} \prod_{g=1}^{k} \frac{\Gamma(\frac{N_g h + n_g}{2})}{\Gamma(\frac{n_g + 1 - i}{2})} \cdot \frac{\Gamma(\frac{N - i}{2})}{\Gamma(\frac{Nh + N - i}{2})} \tag{4.6}$$

For the particular case of $p = 2$, $N_1 = N_2, = \ldots, = N_k = n$, $N = nk$ and $n_g = N_g - 1 = n - 1$ equation (4.6) becomes

$$E(\lambda_0^h) = \frac{(nk)^{knh}}{n^{knh}} \prod_{g=1}^{k} \left[\frac{\Gamma(\frac{nh+n-1}{2}) \; \Gamma(\frac{nh+n-2}{2})}{\Gamma(\frac{n-1}{2}) \qquad \Gamma(\frac{n-2}{2})} \right]$$

$$\frac{\Gamma(\frac{nk-1}{2}) \; \Gamma(\frac{nk-2}{2})}{\Gamma(\frac{nkh+kn-1}{2}) \Gamma(\frac{nkh+kn-2}{2})}$$

$$= k^{knh} \frac{[\Gamma\{\frac{n(h+1)}{2} - \frac{1}{2}\} \; \Gamma\{\frac{n(h+1)}{2} - 1\}]^k}{[\Gamma(\frac{n-1}{2}) \; \Gamma(\frac{n-2}{2})]^k \Gamma\{\frac{nk(h+1)}{2} - \frac{1}{2}\}}$$

$$\cdot \frac{\Gamma(\frac{nk-1}{2}) \; \Gamma(\frac{nk-2}{2})}{\Gamma\{\frac{nk(h+1)}{2} - 1\}} \tag{4.7}$$

Now applying Gauss's multiplication formula (Ref 9:11)

$$\Gamma(mz) = \frac{m^{(mz - \frac{1}{2})}}{(2\pi)^{\frac{m-1}{2}}} \prod_{i=0}^{m-1} \Gamma(z + \frac{i}{m}) \tag{4.7a}$$

with $m = 2$ we can rewrite equation (4.7) in the following simpler form.

$$E(\lambda_0^h) = k^{knh} \frac{\Gamma(kn-2)}{[\Gamma(n-2)]^k} \; \frac{[\Gamma\{n(h+1) - 2\}]^k}{\Gamma\{kn(h+1) - 2\}} \tag{4.8}$$

The h^{th} Moment of λ_1

The h^{th} moment of λ_1 is

$$E(\lambda_1^h) = K_1 \cdot E(L_1^h) \tag{4.9}$$

where $K_1 = \dfrac{N^{\frac{pN}{2}}}{\prod\limits_{g=1}^{k} N_g^{\frac{pN_g}{2}}}$ and $L_1 = \prod\limits_{g=1}^{k} \dfrac{|\underline{A}_g|^{\frac{N_g}{2}}}{|\underline{A}|^{\frac{N}{2}}}$

Since L_1 is a function of $\underline{A}_g$ and $\underline{A}_g \sim W(\underline{A}_g|\underline{\Sigma},n_g)$ we have by definition

$$E(L_1^h) = \int_{\underline{A}_g>0} \left(\prod_{g=1}^{k} |\underline{A}_g|^{\frac{N_g h}{2}}\right) |\underline{A}|^{-\frac{Nh}{2}} \prod_{g=1}^{k} W(\underline{A}_g|\underline{\Sigma},n_g)\, dA_1 \ldots dA_k$$

$$= \prod_{g=1}^{k} K(\underline{\Sigma},n_g) \int_{\underline{A}_g>0} |\underline{A}|^{-\frac{Nh}{2}} \prod_{g=1}^{k} |\underline{A}_g|^{\left(\frac{N_g h + n_g}{2} - \frac{p+1}{2}\right)}$$

$$\cdot \exp\left(-\frac{1}{2} \operatorname{tr} \underline{\Sigma}^{-1}\underline{A}_g\right) dA_1 \ldots dA_k$$

$$= \prod_{g=1}^{k} \frac{K(\underline{\Sigma},n_g)}{K(\underline{\Sigma},N_g h + n_g)} \int_{\underline{A}_g>0} |\underline{A}|^{-\frac{Nh}{2}} \prod_{g=1}^{k} W(\underline{A}_g|\underline{\Sigma},N_g h + n_g)\, dA_1 \ldots dA_k \tag{4.10}$$

The integral on the right hand side of (4.10) is equal to $E(|\underline{A}|^{-\frac{Nh}{2}})$, since $\underline{A} = \underline{A}_1 + \underline{A}_2 + \ldots + \underline{A}_k$.

Therefore,

$$E(L_1^h) = \prod_{g=1}^{k} \frac{K(\underline{\Sigma},n_g)}{K(\underline{\Sigma},N_g h + n_g)} \cdot E(|\underline{A}|^{-\frac{Nh}{2}}) \tag{4.11}$$

And so by theorem 2.3, substituting $-\frac{Nh}{2}$ for h and recalling $\underline{\mathbf{A}} \sim W(\underline{A}|Nh + N - k)$ we have

$$E(|\underline{A}|^{-\frac{Nh}{2}}) = \frac{K(\underline{\Sigma},Nh+N-k)}{K(\underline{\Sigma},Nh+N-k+2(-\frac{Nh}{2}))} = \frac{K(\underline{\Sigma},Nh+N-k)}{K(\underline{\Sigma},N-k)} \tag{4.12}$$

Thus

$$E(\lambda_1^h) = K_1 \left[\prod_{g=1}^{k} \frac{K(\underline{\Sigma},\ n_g)}{K(\underline{\Sigma},N_g h + n_g)}\right]\left[\frac{K(\underline{\Sigma},Nh+N-k)}{K(\underline{\Sigma},N-k)}\right] \tag{4.13}$$

Using the definition of $\Gamma_p(\frac{n}{2})$ and $K(\underline{\Sigma},n)$ from equation (2.13) we have

$$E(\lambda_1^h) = K_1 \prod_{i=1}^{p} \prod_{g=1}^{k} \frac{\Gamma(\frac{n_g + N_g h}{2} + \frac{1-i}{2}) \quad \Gamma(\frac{N-k+1-i}{2})}{\Gamma(\frac{n_g+1-i}{2}) \quad \Gamma(\frac{N-k+1-i+Nh}{2})}$$

where $$K_1 = \frac{N^{\frac{pNh}{2}}}{\prod_{g=1}^{k} N_g^{\frac{pN_g h}{2}}}$$

For the particular case of $p = 2$, $N_1 = N_2 = \ldots$ $N_k = n$, $N = nk$ and $n_g = N_g - 1 = n - 1$, we have

$$E(\lambda_1^h) = k^{knh} \frac{\Gamma\{\frac{k(n-1)}{2}\}\Gamma\{\frac{k(n-1)}{2} - \frac{1}{2}\}}{[\Gamma(\frac{n-1}{2})\,\Gamma(\frac{n-2}{2})]^k}$$

$$\cdot \frac{[\Gamma\{\frac{n(1+h)}{2} - \frac{1}{2}\}\ \Gamma\{\frac{n(1+h)}{2} - 1\}]^k}{\Gamma\{\frac{nk(1+h)}{2} - \frac{1}{2}\}\ \Gamma\{\frac{nk(1+h)}{2} - \frac{k+1}{2}\}} \qquad (4.15)$$

The h^{th} Moment of λ_2

The h^{th} moment of λ_2 is obtained by the same method used to derive λ_0. Thus

$$E(\lambda_2^h) = E(L_2^h) \quad \text{where } L_2 = \left(\frac{|\underline{A}|}{|\underline{A}+\underline{B}|}\right)^{\frac{N}{2}} \qquad (4.16)$$

Since $\underline{A} = \sum_{g=1}^{k} \underline{A}_g$ and each $\underline{A} \sim W(\underline{A}_g|\underline{\Sigma},n_g)$ we have by theorem 2.1 $\underline{A} \sim W(\underline{A}|\underline{\Sigma}, \sum_{g=1}^{k} n_g)$. Now $\sum_{g=1}^{k} n_g = \sum_{g=1}^{k} (N_g - 1)$

$= N - k$ so $\underline{A} \sim W(\underline{A}|\underline{\Sigma},N-k)$. From the discussion of the derivation of λ_0 we know that $\underline{B} \sim W(\underline{B}|\underline{\Sigma}, k-1)$ and $\underline{A}$ and $\underline{B}$ are independent. Since L_2 is a function of $\underline{A}$ and $\underline{B}$ the h^{th} moment of λ_2 is by definition

$$E(L_2^h) = \int_{\underline{A}>0,\underline{B}>0} |\underline{A}|^{\frac{Nh}{2}} |\underline{A}+\underline{B}|^{-\frac{Nh}{2}} K(\underline{\Sigma},N-k)\,|\underline{A}|^{(\frac{N-k}{2} - \frac{p+1}{2})}$$

$$\cdot \exp(-\tfrac{1}{2}\mathrm{tr}\ \underline{\Sigma}^{-1}\underline{A})\ f(\underline{B})\ dA\,dB$$

$$= K(\underline{\Sigma}, N-k) \int_{\underline{A}>0,\underline{B}>0} |\underline{A}+\underline{B}|^{-\frac{Nh}{2}} |\underline{A}|^{(\frac{Nh+N-k}{2}) - \frac{p+1}{2})}$$

$$\exp(-\frac{1}{2} \operatorname{tr} \underline{\Sigma}^{-1}\underline{A}) f(\underline{B})\, dA\, dB$$

$$= \frac{K(\underline{\Sigma}, N-k)}{K(\underline{\Sigma}, Nh+N-k)} \int_{\underline{A}>0,\underline{B}>0} |\underline{A}+\underline{B}|^{-\frac{Nh}{2}} W(\underline{A}|\underline{\Sigma}, Nh+N-k) f(\underline{B})\, dA\, dB \tag{4.17}$$

Recall that $(\underline{A}+\underline{B}) \sim W\{(A+B)|\underline{\Sigma}, Nh+N-1\}$. Therefore, we have

$$E(L_2^h) = \frac{K(\underline{\Sigma}, N-k)}{K(\underline{\Sigma}, Nh+N-k)} \cdot E(|\underline{A}+\underline{B}|^{-\frac{Nh}{2}}) \tag{4.18}$$

Now, applying theorem 2.3, substituting $\underline{A}+\underline{B}$ for $\underline{A}$ and $-\frac{Nh}{2}$ for h we have

$$E(|A+B|^{-\frac{Nh}{2}}) = \frac{K(\underline{\Sigma}, Nh+N-1)}{K(\underline{\Sigma}, Nh+N-1+2(-\frac{Nh}{2}))} = \frac{K(\underline{\Sigma}, nh+N-1)}{K(\underline{\Sigma}, N-1)} \tag{4.19}$$

Combining (4.18) and (4.19) we have

$$E(\lambda_2^h) = \frac{K(\underline{\Sigma}, N-k)}{K(\underline{\Sigma}, Nh+N-k)} \cdot \frac{K(\underline{\Sigma}, Nh+N-1)}{K(\underline{\Sigma}, N-1)} \tag{4.20}$$

Using the definition of $\Gamma_p(\frac{n}{2})$ and $K(\underline{\Sigma}, n)$ for equation (2.13) we have

$$E(\lambda_2^h) = \prod_{i=1}^{p} \frac{\Gamma(\frac{N-k+1-i}{2}+h)\ \Gamma(\frac{N-i}{2})}{\Gamma(\frac{N-k+1-i}{2})\ \ \Gamma(\frac{N-i}{2}+h)} \qquad (4.21)$$

For the particular case of $p = 2$, $N_1 = N_2 = \ldots = N_k = n$ $N = nk$, and $n_g = N_g - 1 = n - 1$, we have

$$E(\lambda_2^h) = \frac{\Gamma(\frac{kn-1}{2})\ \Gamma(\frac{kn-2}{2})}{\Gamma\{\frac{k(n-1)}{2}\}\Gamma\{\frac{k(n-1)}{2}-\frac{1}{2}\}}$$

$$\frac{\Gamma\{\frac{kn(1+h)}{2}-\frac{k}{2}\}\ \Gamma\{\frac{kn(1+h)}{2}-\frac{k+1}{2}\}}{\Gamma\{\frac{kn(1+h)}{2}-\frac{1}{2}\}\ \Gamma\{\frac{kn(1+h)}{2}-1\}} \qquad (4.22)$$

Now, applying Gauss's multiplication formula (4.7a) with $m = 2$ we have

$$E(\lambda_2^h) = \frac{\Gamma(kn-2)\ \Gamma\{kn(1+h)-k-1\}}{\Gamma(kn-k-1)\Gamma\{kn(1+h)-2\}} \qquad (4.23)$$

In this chapter we obtained the h^{th} moment of our test criteria λ_i, $i = 0,1,2$, making extensive use of the Wishart distribution and its properties. The moments were obtained for the general case of p-variables and sample sizes not necessarily equal. We then restricted the moments to $p = 2$ variables and equal sample sizes. Thus, we have the moments of our criteria from which we will obtain their sampling distributions in the next chapter.

V. Distribution of the Criteria

In the literature S. S. Wilks (Ref 17:60) and others have proven that the distribution of $-2 \log \lambda$, where λ is the likelihood ratio criteria, approaches the Chi-Square distribution with r degrees of freedom (r is the number of linear independent restrictions imposed on the null hypothesis), as the sample size n approaches infinity. In this chapter we proceed to obtain the sampling distribution of $-2 \log \lambda_i$, $i = 0,1,2$, by inverting their characteristic functions.

The Distribution of λ_0

The h^{th} moment of λ_0 from equation (4.29) is

$$E(\lambda_0^h) = k^{knh} \frac{\Gamma(nk-2)}{[\Gamma(n-2)]^k} \frac{[\Gamma\{n(h+1)-2\}]^k}{\Gamma\{nk(h+1)-2\}} \tag{5.1}$$

Let $\omega_0 = -2 \log \lambda_0$ and let $\phi_{\omega_0}(t)$ be the characteristic function of ω_0. Then

$$\begin{aligned} \phi_{\omega_0}(t) = E(e^{it\omega_0}) &= E(e^{it(-2\log\lambda_0)}) \\ &= E(e^{\log\lambda_0^{-2it}}) \\ &= E(\lambda_0^{-2it}) \end{aligned} \tag{5.2}$$

Since (5.1) holds for any complex number h, we have substituting -2it for h in equation (5.1)

$$\phi_{\omega_0}(t) = K\, \frac{k^{kn(-2it)}[\Gamma\{n(1-2it)\}-2\}]^k}{\Gamma\{nk(1-2it)-2\}} \tag{5.3}$$

where $$K = \frac{\Gamma(nk-2)}{[\Gamma(n-2)]^k}$$

and therefore

$$\log \phi_{\omega_0}(t) = \log K - (2knit)\log k + k\log [\Gamma\{n(1-2it)-2\}] - \log[\Gamma\{nk(1-2it)-2\}] \tag{5.4}$$

The expansion of $\log \phi_{\omega_0}(t)$ will be based on the following expansion (Ref 1:204):

$$\log \Gamma(x+h) = \frac{1}{2}\log(2\pi) + (x+h-\frac{1}{2})\log x - x - \sum_{r=1}^{m}(-1)^r \frac{B_{r+1}(h)}{r(r+1)x^r} + R_{m+1}(x) \tag{5.5}$$

where $R_m(x)$ is the remainder such that $|R_m(x)| \leq \theta/|x^m|$, θ a constant and $B_r(h)$ is the Bernoulli polynomial of degree r order 1 defined by

$$\frac{\tau e^{h\tau}}{e^{\tau-1}} = \sum_{r=0}^{\infty} \tau\, \frac{B_r(h)}{r!}$$

Extensive tables of Bernoulli polynomials are available in M. A. Fletcher et al. (Ref 4:62-117).

Applying the expansion in (5.5) to the gamma function in (5.4) we have

$$\log \phi_{\omega_0}(t) = \log K - 2\,knit\log k$$

$$+ k \left\{ \frac{1}{2}\log(2\pi) + [n(1-2it) - 2 - \frac{1}{2}] \right.$$

$$\log[n(1-2it)] - n(1-2it)$$

$$\left. - \sum_{r=1}^{m} \frac{(-1)^r B_{r+1}(-2)}{r(r+1)[n(1-2it)]^r} \right\}$$

$$- \left\{ \frac{1}{2}\log(2\pi) + [nk(1-2it) - 2 - \frac{1}{2}] \right.$$

$$\log[nk(1-2it)] - nk(1-2it)$$

$$\left. - \sum_{r=1}^{m} \frac{(-1)^r B_{r+1}(-2)}{r(r+1)[nk(1-2it)]^r} \right\} + R'_{m+1} \qquad (5.6)$$

Let $T = n(1-2it)$ so that $kT = kn(1-2it) = kn - 2knit$. Substituting these results in equation (5.6) we have, after some algebra

$$\log \phi_{\omega_0}(t) = \log[K(2\pi)^{\frac{k-1}{2}} k^{-(kn - 5/2)} T^{-v}]$$

$$+ \sum_{r=1}^{m} \frac{(-1)^r B_{r+1}(-2)}{r(r+1)\ (kT)^r}$$

$$- k \sum_{r=1}^{m} \frac{(-1)^r B_{r+1}(-2)}{r(r+1)T^r} + R'_{m+1} \tag{5.7}$$

where $v = 5/2(k - 1)$

Equation (5.7) can be rewritten as

$$\log \phi_{\omega_0}(t) = \log[K(2\pi)^{\frac{k-1}{2}} k^{-(kn - 5/2)} T^{-v}]$$

$$\exp\left\{\sum_{r=1}^{m} \frac{A_r}{T^r}\right\} + R'_{m+1}(x) \tag{5.8}$$

where $$A_r = \frac{(-1)^r}{r(r+1)} \left[\frac{B_{r+1}(-2)}{k^r} - k\ B_{r+1}(-2)\right] \tag{5.9}$$

Thus, from equation (5.7) we have

$$\phi_{\omega_0}(t) = K(2\pi)^{\frac{k-1}{2}} k^{-(kn - 5/2)} T^{-v}$$

$$\left[\sum_{i=0}^{\infty} \frac{Q_i}{T^i}\right] + R_{m+1}(x) \tag{5.10}$$

The coefficients Q_i are recursively computed using the following relation between A_i and Q_i

$$Q_i = \sum_{k=1}^{i} k \frac{A_k Q_{i-k}}{i}, \quad Q_0 = 1 \tag{5.11}$$

(Ref 6: Chapters 4, 5)

Recalling that $T = n(1 - 2it)$, we have from (5.9) the characteristic function of ω_0 as

$$\phi_{\omega_0}(t) = K(2\pi)^{\frac{k-1}{2}} k^{-(kn - 5/2)} \sum_{r=0}^{m} Q_r [n(1 - 2it)]^{\frac{-2(v+r)}{2}} + R_{m+1}(x) \tag{5.12}$$

$(1 - 2it)^{-\frac{2(v+r)}{2}}$ is the characteristic function of a Chi-Square variable with $2(v+r)$ degrees of freedom. We have, on inverting the characteristic function of ω_0 in (5.11), the p.d.f. of ω_0 is

$$f(\omega_0) = K(2\pi)^{\frac{k-1}{2}} k^{-(kn - 5/2)} \sum_{r=0}^{m} Q_r \left(\frac{1}{n}\right)^{v+r} \chi^2_{2(v+r)} + R_{M+1}(x) \tag{5.13}$$

where $Q_0 = 1$ and $K = \dfrac{\Gamma(nk-2)}{[\Gamma(n-2)]^k}$, $V = \frac{5}{2}(k - 1)$

Thus we have the following theorem:

Theorem 5.1: Under the null hypothesis in (3.1i), the distribution of $\omega_0 = -2 \log \lambda_0$ can be represented as the following linear combinations of Chi-Square distributions:

$$P(\omega_0 \geq x) = K(2\pi)^{\frac{k-1}{2}} k^{-(kn-5/2)} \sum_{r=0}^{\infty} n^{-(v+r)}$$

$$Q_r \, P(\chi^2_{2(v+r)} \geq x) \tag{5.14}$$

where $v = 5/2(k-1)$ and $K = \dfrac{\Gamma(nk-2)}{[\Gamma(n-2)]^k}$

The Distribution of λ_1

The h^{th} moment of λ_1 is given by equation (4.21) as

$$E(\lambda_1^h) = K_1 k^{knh} \frac{[\Gamma\{\frac{n}{2}(1+h) - \frac{1}{2}\}]^k [\Gamma\{\frac{n}{2}(1+h) - 1\}]^k}{\Gamma\{\frac{kn}{2}(1+h) - \frac{k}{2}\} \quad \Gamma\{\frac{kn}{2}(1+h) - \frac{1}{2} - \frac{k}{2}\}} \tag{5.15}$$

where $$K_1 = \frac{\Gamma\{\frac{k(n-1)}{2}\}\Gamma\{\frac{k(n-1)}{2} - \frac{1}{2}\}}{[\Gamma(\frac{n-1}{2}) \quad \Gamma(\frac{n-2}{2})]^k}$$

Let $\omega_1 = -2 \log \lambda_1$ and $\phi_{\omega_1}(t)$ be the characteristic function of ω_1. Then

$$\phi_{\omega_1}(t) = E(\lambda^{-2it}) = K_1\, k^{kn(-2it)}$$

$$\frac{[\Gamma\{\frac{n}{2}(1-2it) - \frac{1}{2}\}\ \Gamma\{\frac{n}{2}(1-2it) - 1\}]^k}{\Gamma\{\frac{kn}{2}(1-2it) - \frac{k}{2}\}\ \Gamma\{\frac{kn}{2}(1-2it) - \frac{1}{2} - \frac{k}{2}\}} \tag{5.16}$$

Taking the log $\phi_{\omega_1}(t)$, applying the gamma expansion formula (5.5) and letting $T = \frac{n}{2}(1 - 2it)$ we have

$$\log \phi_{\omega_1}(t) = \log K_1 - 2itkn \log k + (a) + (b) + (c) + (d) \tag{5.17}$$

where

$$(a) = \frac{k}{2}\log(2\pi) + k[(T-1)\log T - T]$$

$$- k \sum_{r=1}^{m} \frac{(-1)^r B_{r+1}(-\frac{1}{2})}{r(r+1)T^r} + k R^a_{m+1}(T)$$

$$(b) = \frac{k}{2}\log(2\pi) + k[(T-\frac{3}{2})\log T - T]$$

$$- k \sum_{r=1}^{m} \frac{(-1)^r B_{r+1}(-1)}{r(r+1)T^r} + k R^b_{m+1}(kT)$$

$$(c) = -\frac{1}{2}\log(2\pi) - [(kT - \frac{k}{2} - \frac{}{2})\log kT - kT]$$

$$+ \sum_{r=1}^{m} \frac{(-1)\ B_{r+1}(-\frac{b}{2})}{r(r+1)(kT)^r} + R^c_{m+1}(kT)$$

$$(d) = -\frac{1}{2}\log(2\pi) - [(kT - \frac{k}{2} - 1)\log kT - kT]$$

$$+ \sum_{r=1}^{m} \frac{(-1)^r B_{r+1}(-\frac{(k+1)}{2})}{r(r+1)(kT)^r} + R^d_{m+1}(kT)$$

After some algebra, equation (5.17) reduces to

$$\log \phi_{\omega_1}(t) = \log K_1 + (k-1)\log(2\pi) + [k(1-n) + \frac{3}{2}]$$

$$\log k + [\frac{3(1-k)}{2}] \log T$$

$$+ \sum_{r=1}^{m} \frac{A_r}{T^r} + R'_{m+1}$$

$$= \log [K_1 (2\pi)^{k-1} k^{[k(1-r)+3/2]} T^{-v}$$

$$\exp \left\{ \sum_{r=1}^{m} \frac{A_r}{T^r} \right\} + R'_{m+1} \tag{5.18}$$

where $$A_r = \frac{(-1)^r}{r(r+1)} \left[\frac{B_{r+1}(-\frac{k}{2})}{k^r} + \frac{B_{r+1}(-\frac{(k+1)}{2})}{k^r} \right.$$

$$\left. - k\, B_{r+1}(-1) - k\, B_{r+1}(-\frac{1}{2}) \right]$$

and $$v = \frac{3(k-1)}{2}.$$

Thus we have from (5.18)

$$\phi_{\omega_1}(t) = K_1 (2\pi)^{k-1} k^{[k(1-n)+3/2]} T^{-v}$$

$$\left[\sum_{i=0}^{\infty} \frac{Q_i}{T^i}\right] + R_{m+1} \tag{5.19}$$

where the coefficients Q_i can be recursively computed as in (5.11).

Recalling that $T = \frac{n}{2}(1 - 2it)$, we have from equation (5.19) the characteristic function of ω_1 as

$$\phi_{\omega_1}(t) = K_1 (2\pi)^{k-1} k^{(k-kn+3/2)}$$

$$\sum_{r=0}^{m} Q_r \left[\frac{n}{2}(1-2it)\right]^{\frac{-2(v+r)}{2}} + R_{m+1} \tag{5.20}$$

Since $(1-2it)^{\frac{-2(v+r)}{2}}$ is the characteristic function of a Chi-Square density with $2(v+r)$ degrees of freedom, we have on inverting the characteristic function of ω_1 in (5.12) the p.d.f. of ω_1 as

$$f(\omega_1) = K_1 (2\pi)^{k-1} k^{(k-kn+3/2)}$$

$$\sum_{r=0}^{m} Q_r \left(\frac{2}{n}\right)^{v+r} \chi^2_{2(v+r)} + R_{m+1}$$

and thus we have the following theorem:

Theorem 5.2: Under the null hypothesis in (3.1ii), the distribution of $\omega_1 = -2 \log \lambda_1$ can be represented as the following linear combinations of Chi-Square distributions:

$$P(\omega_1 \geq x) = K_1 (2\pi)^{k-1} k^{(k - kn + 3/2)}$$

$$\sum_{r=0}^{\infty} \left(\frac{2}{n}\right)^{v+r} Q_r P(\chi^2_{2(v+r)} \geq x) \qquad (5.21)$$

where $\quad v = \frac{3}{2}(k - 1)$ and $K_1 = \dfrac{\Gamma\{\frac{k(n-1)}{2}\}\Gamma\{\frac{k(n-1)}{2} - \frac{1}{2}\}}{[\{(\frac{n-1}{2}) \quad \Gamma(\frac{n-2}{2})\}]^k}$.

The Distribution of λ_2

The h^{th} moment of λ_2 from equation (4.23) is

$$E(\lambda_2^h) = \frac{\Gamma(nk-2)\,\Gamma\{nk(1+h) - k - 1\}}{\Gamma(nk - k - 1)\Gamma\,\{nk(1+h) - 2\}} \qquad (5.22)$$

Let $L_2 = \lambda_2^{1/N}$ where $N = nk$. We then have from (5.22)

$$E(L_2^h) = \frac{\Gamma(nk-2)}{\Gamma(nk-k-1)} \frac{\Gamma\{nk(1+\frac{h}{nk}) - k - 1\}}{\Gamma\{nk(1+\frac{h}{nk}) - 2\}}$$

$$= \frac{\Gamma(nk-2)}{\Gamma(nk-k-1)} \frac{\Gamma\{k(n-1) - 1 + h\}}{\Gamma(nk - 2 + h)} \qquad (5.23)$$

Note that the h^{th} moment of a beta distribution is

$$\frac{\Gamma\{\frac{1}{2}(a+b)\}\ \Gamma\{\frac{1}{2}(a+h)\}}{\Gamma\{\frac{1}{2}(a+b)+h\}\Gamma(\frac{1}{2})^a}$$

with parameters a/2 and b/2 (Ref 1:194).

Thus from (5.23) we see that L_2 has a beta p.d.f. with parameters $N - k - 1$ and $k - 1$. So we have the following theorem:

Theorem 5.3: Under the null hypothesis in (3.1iii) the distribution of L_2 is given by

$$P(L_2 \leq x) = I_x(N - k - 1,\ k - 1) \tag{5.24}$$

where $I_x(.,.)$ is the incomplete beta function.

Numerical Computations

The c.d.f. of $\omega_i = -2 \log \lambda_i$ $(i = 0,1)$ given in equations (5.14) and (5.21) are used to compute percentage points of ω_i at the level of significance $\alpha = .01$ and $\alpha = .05$ with sample size n from 3 to 100 and $k = 2(1)6$. These are presented in Tables IV and V in the appendix. Tables I, II and III in the appendix give the percentage points of L_0, L_1, and L_2 respectively. The following considerations are taken in checking the accuracy of the computations in the percentage points:

1. The integral over zero to infinity of the c.d.f.'s in (5.14) and (5.21) rapidly approaches one. Table 5-1 for k = 6, Theorem 5.2, gives the typical behavior of the series as the number of terms increases. To achieve accuracy to five significant figures in all cases considered required fifteen terms; and

2. The exact values are in close agreement with the approximate values obtained, using the asymptotic expansion in Chapter VI, even for comparatively small values of n.

TABLE 5-1

EVALUATION OF c.d.f. OF THEOREM 5.2 for m TERMS

m	n = 10	n = 20
1	.2160599	.4839011
2	.5146427	.8182633
3	.7491630	.9495748
4	.8868061	.9881090
5	.9539563	.9975087
6	.9827359	.9995229
7	.9939364	.9999149
8	.9979812	.9999857
9	.9993571	.9999977
10	.9998028	.9999996
11	.9999414	.9999999
12	.9999831	1.0000000
13	.9999952	1.0000000
14	.9999987	1.0000000
15	.9999996	1.0000000

Summary

In this chapter we obtained the sampling distributions of our test criteria λ_0, λ_1 and λ_2. Note that the distributions of both λ_0 and λ_1 are linear combinations of Chi-Square distributions. The distribution of λ_2 resulted

in a beta distribution with parameters $k(n-1)-1$ and $k-1$. From the distributions, percentage points for different sample sizes and k populations can be computed.

Providing tables for numerous populations and various sample sizes would be inconvenient and time-consuming; therefore, a good approximation for moderate sample sizes would be extremely beneficial. It is this topic that we treat in the next chapter.

VI. Asymptotic Approximation to the Distribution

Although the tables of percentage points fill a gap and meet some of the needs of statisticians, approximations to the distributions of the various criteria are the only practical means for computing the observed significance probabilities in the analysis.

In this chapter we develop an asymptotic expansion to the distributions of λ_0 and λ_1 with the second term of the order m^{-2} so that the first term alone should provide a powerful approximation to the percentage points of λ_0 and λ_1 even for comparatively small values of n.

Asymptotic Expansion of the Distribution of λ_0

Let $M_0 = -2\rho \log \lambda_0$ where ρ is an arbitrary constant to be chosen later. From equation (5.1), substituting $-2\rho it$ for h, the characteristic function of M_0 is given by

$$\phi_{M_0}(t) = E(\lambda_0^{-2\rho it}) = K_0\, C(t) \tag{6.1}$$

where

$$K_0 = \frac{\Gamma(nk-2)}{[\Gamma(n-2)]^k} \tag{6.1a}$$

and

$$C(t) = k^{-2\rho itkn} \frac{[\Gamma\{\rho n(1-2it) + (n-\rho n-2)\}]^k}{\Gamma\{\rho nk(1-2it) + (nk-\rho nk-2)\}} \quad (6.1b)$$

Let $T = m(1-2it)$ where $m = \rho n$. Then we have

$$\log C(t) = (Tk - \rho nk) \log k + k \log \Gamma\{T + (n-\rho n-2)\}$$
$$- \log\Gamma\{kT + (nk-\rho nk-2)\} \quad (6.2)$$

Using the asymptotic expansion formula (5.5) in (6.2), we have the asymptotic expansion of $C(t)$.

$$\log C(t) = (-nk + \frac{5}{2}) \log k + \frac{(k-1)}{2} \log(2\pi) - v \log T$$
$$+ \sum_{r=1}^{u} \frac{A_r}{T^r} + R'_{u+1} \quad (6.3)$$

where $v = \frac{5}{2}(k - 1)$

and $$A_r = \frac{(-1)^r}{r(r+1)k^r} [B_{r+1}(nk - \rho nk - 2) - k^{r+1}$$
$$B_{r+1}(n - \rho n - 2)] \quad (6.3a)$$

Thus the asymptotic expansion of $C(t)$ up to the order m^{-2} is given by

$$C(t) = (2\pi)^{(\frac{k-1}{2})} k^{-(kn - \frac{5}{2})} T^{-v}$$
$$[1 + \frac{Q_1}{T} + \frac{Q_2}{T^2} + O(m^{-3})] \quad (6.4)$$

where $Q_1 = A_1$ and $Q_2 = A_2 + \frac{A_1^2}{2}$.

Now from equation (6.1a) we have

$$\log K_0 = \log \Gamma\{mk + (nk - \rho nk - 2)\}$$

$$- k \log \Gamma(m + n - \rho n - 2) \qquad (6.5)$$

Again using the asymptotic expansion (5.5) to (6.5) we have after some simplification

$$\log K_0 = \left(\frac{1-k}{2}\right) \log(2\pi) + v \log m + (nk - \tfrac{5}{2}) \log k$$

$$+ \sum_{r=1}^{u} \frac{A'_r}{m^r} + R''_{u+1} \qquad (6.6)$$

where $A'_r = -A_r$ and $v = \frac{5}{2}(k-1)$ and so

$$K_0 = (2\pi)^{\left(\frac{1-k}{2}\right)} k^{(nk-5/2)} m^v \left[1 + \frac{Q'_1}{m} + \frac{Q'_2}{m^2} + O(m^{-3})\right] \qquad (6.7)$$

where $Q'_1 = -A_1$ and $Q'_2 = (-A_2) + \frac{(-A_1)^2}{2}$.

We now choose ρ such that $A_1 = 0$. Then

$$\rho = 1 - \frac{37(k+1)}{30\,nk} \qquad (6.8)$$

From (6.1), (6.4) and (6.7) the characteristic function of M_0 is given by

$$\phi_{M_0}(t) = (1-2it)^{-v}\left[1 + \frac{Q_2}{m^2(1-2it)^2} + O(m^{-3})\right]$$

$$\left[1 + \frac{Q'_2}{m^2(1-2it)} + O(m^{-3})\right]$$

$$= (1-2it)^{-v} + \frac{Q_2}{m^2}\left[(1-2it)^{-v-2} - (1-2it)^{-v}\right]$$

$$+ O(m^{-3}) \qquad (6.9)$$

Therefore, inverting the characteristic function (6.9), we have the following theorem:

Theorem 6.1: Under the hypothesis (3.1i) the asymptotic expansion of $M_0 = -2\rho \log \lambda_0$ up to the order m^{-2} is given by

$$P(M_0 \geq x) = P(\chi^2_{2v} \geq x) + \frac{Q_2}{m^2}$$

$$\left[P(\chi^2_{2(v+2)} \geq x) - P(\chi^2_{2v} \geq x)\right] + O(m^{-3}) \qquad (6.10)$$

where $v = \frac{5}{2}(k-1)$ $m = \rho n$ and ρ is as in (6.8).

Remark: Since the second term m (6.10) is of the order m^{-2}, the first term alone provides a powerful approximation to the percentage points of λ_0 as seen from Table 6-1 and Table 6-2.

TABLE 6-1

COMPARISON OF APPROXIMATION AND EXACT DISTRIBUTION
$\alpha = .01$

n \ k	2	3	4	5	6
3	.04028* .03763	.06222 .05775	.08586 .07256	.10960 .08432	.13324 .09378
4	.16708 .17302	.18264 .19377	.20110 .21094	.21897 .22498	.23597 .23630
5	.29807 .30197	.30775 .31600	.32173 .33081	.33460 .34372	.34631 .35433
10	.62902 .62951	.62832 .62962	.63505 .63617	.64194 .64325	.64805 .64948
15	.75051 .75063	.74821 .74868	.75236 .75257	.75696 .75726	.76115 .76151
25	.84963 .84965	.84730 .84746	.84956 .84952	.85229 .85229	.85483 .85486
50	.92466 .92465	.92311 .92317	.92414 .92409	.92549 .92545	.92676 .92674
75	.94975 .94974	.94863 .94867	.94929 .94925	.95018 .95015	.95103 .95101
100	.96230 .96230	.96143 .96146	.96192 .96189	.96259 .96256	.96322 .96321

*The upper number is the exact value $L_0 = \lambda_0^{1/kn}$.

The lower number is the approximation value.

TABLE 6-2

COMPARISON OF APPROXIMATION AND EXACT DISTRIBUTION
$\alpha = .05$

n\k	2	3	4	5	6
3	.08728* .09006	.10343 .10526	.12627 .11732	.15082 .12658	.17639 .13395
4	.26961 .27594	.26141 .27374	.26769 .28050	.27759 .28746	.28911 .29365
5	.41201 .41527	.39565 .40273	.39639 .40511	.40047 .40963	.40560 .41425
10	.71171 .71201	.69325 .69402	.69007 .69111	.69044 .69161	.69185 .69310
15	.81011 .81016	.79553 .79572	.79251 .79279	.79234 .79265	.79306 .79340
25	.88730 .88729	.87748 .87751	.87523 .87528	.87491 .87496	.87523 .87529
50	.94414 .94413	.93884 .93884	.93755 .93755	.93730 .93730	.93742 .93742
75	.96287 .96286	.95926 .95925	.95836 .95835	.95817 .95817	.95824 .95823
100	.97220 .97219	.96945 .96945	.96876 .96876	.96862 .96861	.96867 .96866

*The upper number is the exact value $L_0 = \lambda_0^{1/kn}$.

The lower number is the approximation value.

Asymptotic Expansion of the Distribution of λ_1

Let $M_1 = -2q \log \lambda_1$, where q is an arbitrary constant to be chosen later. From equation (5.15) substituting $-2\,qit$ for h the characteristic function of M_1 is given by

$$\phi_{M_1}(t) = K_1 C_1(t) \tag{6.11}$$

where
$$K_1 = \frac{\Gamma\{\frac{k(n-1)}{2}\}\,\Gamma\{\frac{k(n-1)}{2} - \frac{1}{2}\}}{[\Gamma(\frac{n-1}{2}) \quad \Gamma(\frac{n-2}{2})]^k} \tag{6.11a}$$

and
$$C_1(t) = k^{-2\,qit\,kn} \frac{[\Gamma\{\frac{n}{2}(1-2\,qit) - \frac{1}{2}\}]^k}{\Gamma\{\frac{nk}{2}(1-2\,qit) - \frac{k}{2}\}}$$

$$\frac{[\Gamma\{\frac{n}{2}(1-2\,qit) - 1\}]^k}{\Gamma\{\frac{nk}{2}(1-2\,qit) - \frac{k+1}{2}\}} \tag{6.11b}$$

Let $T = \frac{qn}{2}(1 - 2\,it)$ and applying the asymptotic expansion formula (5.5) to the $\log C_1(t)$ in (6.11b) we have after some algebra

$$\log C_1(t) = (k-1)\log(2\pi) + (-kn + k + \tfrac{3}{2})\log k - v \log T$$

$$+ \sum_{r=1}^{u} \frac{A_r}{T^r} + R'_{u+1} \tag{6.12}$$

where $\quad v = \frac{3}{2}(k-1)$

and $$A_r = \frac{(-1)^r}{r(r+1)k^r} [B_{r+1}(\frac{kn-k-qkn}{2})$$

$$+ B_{r+1}(\frac{kn-k-qkn-1}{2})$$

$$- k^{r+1}\{B_{r+1}(\frac{n-qn-1}{2})$$

$$+ B_{r+1}(\frac{n-qn-2}{2})\}] \qquad (6.12a)$$

Thus the asymptotic expansion of $C_1(t)$ up to the order m^{-2} is given by

$$C_1(t) = (2\pi)^{(k-1)} k^{(-kn+k+\frac{3}{2})} T^{-v} [1 + \frac{Q_1}{T} + \frac{Q_2}{T^2} + O(m^{-3})] \qquad (6.13)$$

where $Q_1 = A_1$ and $Q_2 = A_2 + \frac{A_1^2}{2}$.

Now from equation (6.11a) where $m = \frac{qn}{2}$ we have

$$K_1 = \frac{\Gamma(km + \frac{kn-k-qkn}{2})\,\Gamma(km + \frac{kn-k-1-qkn}{2})}{[\Gamma(m + \frac{n-qn-1}{2}) \quad \Gamma(m + \frac{n-qn-2}{2})} \qquad (6.14)$$

Again applying the asymptotic expansion formula (5.5) to $\log K_1$ in (6.14) we have

$$\log K_1 = (1-k)\log(2\pi) + v\log m + (kn-k-\frac{3}{2})\log k$$

$$+ \sum_{r=1}^{u} \frac{A'_r}{m^r} + R''_{u+1} \qquad (6.15)$$

where $A'_r = -A_r$.

And so the asymptotic expansion of K_1 up to the order m^{-2} is given by

$$K_1 = (2\pi)^{(k-1)} \; m^v \, k^{(kn - k - \frac{3}{2})} \left[1 + \frac{Q'_1}{m} + \frac{Q'_2}{m^2} + O(m^{-3})\right] \tag{6.16}$$

where $A'_1 = -A_1$ and $Q'_2 = -A_2 + \frac{(-A_1)^2}{2}$

Now choose q such that $A_1 = 0$. Then

$$q = 1 - \frac{31k + 13}{18\,nk} \tag{6.17}$$

From (6.11), (6.13) and (6.16) the characteristic function of M_1 is given by

$$\phi_{M_1}(t) = (1 - 2\,it)^{-v} \left[1 + \frac{Q_2}{m^2(1 - 2\,it)^2} + O(m^{-3})\right]$$

$$\left[1 + \frac{Q'_2}{m^2(1 - 2\,it)^2} + O(m^{-3})\right]$$

$$= (1 - 2\,it)^{-v} + \frac{Q_2}{m^2}\left[(1 - 2\,it)^{-v-2} - (1 - 2\,it)^{-v}\right]$$

$$+ O(m^{-3}) \tag{6.18}$$

Therefore, inverting the characteristic function $\phi_{M_1}(t)$ in (6.18) we have the following theorem:

Theorem 6.2: Under the hypothesis (3.1ii) the asymptotic expansion of $M_1 = -2q \log \lambda_1$ up to order m^{-2} is given by

$$P(M_1 \geq x) = P(\chi^2_{2v} \geq x) + \frac{Q_2}{m^2} [P(\chi^2_{2(v+2)} \geq x) - P(\chi^2_{2v} \geq x)] + O(m^{-3}) \tag{6.19}$$

Remark: The second term in (6.19) is of the order of m^{-2} and so the first term alone should provide a good approximation to the percentage points of λ_1, even for relatively small sample sizes as shown in Tables 6-3 and 6-4.

TABLE 6-3

COMPARISON OF APPROXIMATION AND EXACT DISTRIBUTION
α = .01

n \ k	2	3	4	5	6
3	.06038 .04557	.09887 .06709	.12372 .08424	.15609 .09869	.18748 .11041
4	.22992 .22829	.25183 .25274	.27559 .27509	.29787 .29221	.31870 .30661
5	.37893 .37882	.39586 .39752	.41467 .41731	.43083 .43273	.44484 .44585
10	.69895 .69934	.70546 .70567	.71541 .71585	.72409 .72419	.73131 .73150
15	.80287 .80317	.80655 .80662	.81311 .81329	.81894 .81886	.82382 .82379
25	.88359 .88378	.88547 .88549	.88934 .88942	.89284 .89274	.89578 .89571
50	.94253 .94263	.94334 .94334	.94525 .94528	.94699 .94693	.94846 .94847
75	.96185 .96192	.96236 .96237	.96363 .96365	.96479 .96475	.96577 .96574
100	.97145 .97150	.97182 .97183	.97277 .97278	.97364 .97361	.97437 .97435

*The upper number is the exact value $L_1 = \lambda_1^{1/kn}$.

The lower number is the approximation value.

TABLE 6-4

COMPARISON OF APPROXIMATION AND EXACT DISTRIBUTION
$\alpha = .05$

n \ k	2	3	4	5	6
3	.13239 .11862	.15090 .13208	.17995 .14541	.21108 .15637	.24261 .16530
4	.36242 .36075	.35480 .35680	.36258 .36466	.37372 .37315	.38636 .38073
5	.51231 .51171	.49968 .50095	.50294 .50506	.50869 .51111	.51484 .51695
10	.78134 .78126	.77006 .77012	.76992 .77006	.77192 .77216	.77434 .77462
15	.85964 .85960	.85128 .85127	.85083 .85084	.85199 .85204	.85350 .85356
25	.91828 .91826	.91292 .91290	.91249 .91247	.91311 .91310	.91396 .91397
50	.96005 .96004	.95725 .95724	.95698 .95697	.95726 .95725	.95767 .95766
75	.97356 .97356	.97168 .97167	.97148 .97147	.97166 .97165	.97193 .97192
100	.98025 .98024	.97882 .97881	.97867 .97866	.97880 .97880	.97900 .97900

*The upper number is the exact value $L_1 = \lambda_1^{1/kn}$.

The lower number is the approximation value.

VII. Practical Illustration

The Engineering and Design Group, AFWAL/MLSE, Wright-Patterson AFB, Ohio, provided the data used in the following analysis.

The ultimate tensile strength of a metal alloy is characterized by two correlated variables, x and y, longitudinal hardness and transversal hardness, respectively.

Test procedures:

1. Three companies received a sample block of metal alloy IN-9021, hand forged.

2. Each company used a common set of test conditions and conducted identical tests in accordance with ASTM testing standards.

3. Three measurements of longitudinal hardness and three measurements of transversal hardness were obtained by each company.

The collected data is as shown in Table 7-1, by company, where L represents longitudinal hardness ksi and T represents transversal hardness ksi.

We shall first test H_0, the hypothesis that there is no significant difference between the populations. A summary of the necessary calculations is shown in Table 7-2, where the number of populations, k = 3, and the number of observations, n = 3.

TABLE 7-1

TENSILE STRENGTH: VARIABLES L AND T

General Dynamics		Lockheed		Rockwell	
L	T	L	T	L	T
87.3	85.9	89.0	87.4	87.6	86.0
84.4	86.0	89.6	87.3	86.3	84.7
89.8	84.8	89.6	87.3	83.9	84.7

TABLE 7-2

SUMMARY OF CALCULATIONS

$\lvert \underline{A}_1 \rvert$	$\lvert \underline{A}_2 \rvert$	$\lvert \underline{A}_3 \rvert$	$\lvert \underline{A} \rvert$	$\lvert \underline{A}+\underline{B} \rvert$
2.88115	.00120	3.24480	43.1704	282.74248

From equation (3.15) with $L_0^* = \lambda_0^{1/kn}$ we have

$$L_0^* = k \frac{\left[\prod_{g=1}^{k} |\underline{A}_g|\right]^{\frac{1}{2k}}}{|\underline{A} + \underline{B}|^{\frac{1}{2}}} = 3 \frac{\left[\prod_{g=1}^{3} |\underline{A}_g|\right]^{\frac{1}{6}}}{|\underline{A} + \underline{B}|^{\frac{1}{2}}} = .0836$$

From Table I in the appendix with $\alpha = .05$, $n = 3$, $k = 3$, $L_0 = .10343$.

Decision Rule: Reject H_0 if $L_0^* \leq L_{\alpha,n,k}$.

Since $L_0^* < .10343$, we reject H_0 at the $\alpha = .05$ significance level and conclude the populations as regard to tensile strength are not identical.

We now proceed to test H_1, the hypothesis that there is no significant difference in the populations as regards variance and covariance in the variables x and y.

From equation (3.16) with $L_1^* = \lambda_1^{1/kn}$ we have

$$L_1^* = k \prod_{g=1}^{k} \frac{[|\underline{A}_g|]^{\frac{1}{2k}}}{|\underline{A}|^{\frac{1}{2}}} = 3 \frac{\prod_{g=1}^{3} |\underline{A}_g|]^{\frac{1}{6}}}{|\underline{A}|^{\frac{1}{2}}} = .2087$$

From Table II in the appendix with $\alpha = .05$, $n = 3$, $k = 3$; $L_1 = .1509$.

By the Decision Rule, since $L_1^* > .1509$ we do not reject H_1, that the variances and covariance are the same at $\alpha = .05$ significance level.

We may further test H_2, the hypothesis that the means are the same among the populations given that the variances and covariances are equal.

From equation (3.17) with $L_2^* = \lambda_2^{1/kn}$ we have

$$L_2^* = \left[\frac{|\underline{A}|}{|\underline{A}+\underline{B}|}\right]^{\frac{1}{2}} = .3907$$

From Table III in the appendix with $\alpha = .05$, $n = 3$, $k = 3$, $L_2 = .4182$.

The Decision Rule leads us to reject H_2 and conclude that the means among the populations are not equal. This is consistent with our previous conclusion concerning H_0.

In summary, the conclusions are, the sample ingots of metal alloys received by the companies have equal variances and covariances concerning the two attributes, longitudinal and transversal hardness, but the means differ at the 5 percent significance level.

VIII. Conclusion

In this thesis we obtained the exact distribution for our test criteria λ_0, λ_1 and λ_2. To do this we assumed that our samples were drawn from populations distributed as $N_p(\underline{x}|\underline{u}^g,\underline{\Sigma}_g)$ and then restricted the development of the sampling distributions to the case of equal sample sizes and two variables. From the exact distributions we were able to obtain tables of percentage points whic.. enables one to test the hypotheses considered in this thesis.

The asymptotic approximations to the distribution of λ_0 and λ_1 extended our testing ability to sample sizes and populations not covered by tables. Tables of comparisons in Chapter VI showed that the asymptotic expansion yields powerful approximations to the percentage points of the test statistics.

The importance of multivariate analysis is illustrated by the many entities that require several traits to describe their characteristics. Testing all the attributes simultaneously is necessary because multiple correlations may exist among the variables. For example, the quality of a relay might be accurately characterized by three variables; capacitance, inductance and resistance, a metal alloy may require the variables; shear strength and

compression strength in addition to tensile strength to adequately describe its quality.

The application of this theory to practical problems is unlimited including areas such as agriculture, anthropology, economics, physics, industry, medicine and sociology, to name a few.

In light of this, areas of further study include extending multivariate methods to more than two variables and unequal sample sizes in obtaining the sampling distributions. Also, in order to study the power of the tests it would be worthwhile to develop methods to obtain the non-null distributions of the criteria.

Bibliography

1. Anderson, T. W. An Introduction to Multivariate Statistical Analysis. New York: John Wiley and Sons, Inc., 1958.

2. Chung, Kailai. A Course in Probability Theory. New York: Academic Press, 1968.

3. Erdelyi, A., et al. Higher Transcendental Functions, Volume I. New York: McGraw-Hill, 1953.

4. Fletcher, M. A. An Index of Mathematical Tables, Volume I. Boston: Addison-Wesley Publishing Company, Inc., pp. 62-117 (1962).

5. Hogg, R. V. and A. T. Craig. Introduction to Mathematical Statistics (Fourth edition). New York: Macmillan Publishing Co., Inc., 1978.

6. Kalinin, V. M. and O. V. Shalaevskii. Investigations in Classical Problems of Probability Theory and Mathematical Statistics, Part I. New York: Consultants Bureau (1971).

7. Kendall, M. G. and A. Stuart. The Advanced Theory of Statistics, Volume 3. New York: Hafner Publishing Company, 1966.

8. Kshirsagar, A. M. Multivariate Analysis. New York: Marcel Decker, Inc., 1972.

9. Luke, Y. L. The Special Functions and Their Approximations, Volume I. New York: Academic Press, 1979.

10. Nagarsenker, B. N. and K. C. S. Pillai. "The Distribution of the Sphericity Test Criterion," Aerospace Research Laboratories 72-0154 (November 1972).

11. Neyman, J. and E. S. Pearson. "On the Problem of Two Samples," Neyman and Pearson: Joint Statistical Papers, edited by the Trustees of Biometrika. Los Angeles: University of California Press (1966).

12. Neyman, J. and E. S. Pearson. "On the Problem of k Samples," Neyman and Pearson: Joint Statistical Papers, edited by the Trustees of Biometrika. Los Angeles: University of California Press (1966).

13. Pearson, E. S. and S. S. Wilks. "Methods of Statistical Analysis Appropriate for k Samples of Two Variables," Biometrika, 25: 353-378 (1933).

14. Romanovsky, V. "On Criteria that Two Given Samples Belong to the Same Normal Population," Metron, 7: 3-46 (1969).

15. Srivastava, M. S. and C. G. Khatri. An Introduction to Multivariate Statistics. New York: McGraw-Hill, 1979.

16. Wilks, S. S. "Test Criteria for Statistical Hypothesis Involving Several Variables," Journal of American Statistical Assoc., 30: 549-560 (1935).

17. Wilks, S. S. "The Large Sample Distribution of the Likelihood Ratio for Testing Composite Hypotheses," Ann. Math Statist., 9: 60-62 (1938).

Appendix

Tables

Table I. Percentage Points of L_0

$\alpha = .01$

N	2	3	4	5	6
3	.04028	.06222	.08586	.10960	.13324
4	.16708	.18264	.20110	.21897	.23597
5	.29807	.30775	.32173	.33460	.34631
6	.40062	.40647	.41823	.42903	.43844
7	.47927	.48246	.49253	.50204	.51029
8	.54056	.54194	.55065	.55912	.56653
9	.58937	.58952	.59713	.60474	.61144
10	.62902	.62832	.63505	.64194	.64805
11	.66181	.66053	.66654	.67281	.67842
12	.68935	.68766	.69308	.69883	.70401
13	.71278	.71081	.71573	.72105	.72585
14	.73296	.73080	.73530	.74023	.74471
15	.75051	.74821	.75236	.75696	.76115
16	.76590	.76353	.76736	.77167	.77561
17	.77952	.77709	.78066	.78471	.78842
18	.79165	.78919	.79252	.79635	.79986
19	.80251	.80005	.80318	.80679	.81012
20	.81231	.80985	.81279	.81622	.81939
21	.82118	.81874	.82151	.82477	.82779
22	.82925	.82683	.82946	.83257	.83545
23	.83663	.83424	.83673	.83970	.84246
24	.84340	.84103	.84340	.84625	.84890
25	.84963	.84730	.84956	.85229	.85483
30	.87459	.87244	.87427	.87653	.87866
35	.89245	.89048	.89201	.89395	.89577
40	.90586	.90405	.90537	.90706	.90865
45	.91630	.91463	.91579	.91728	.91870
50	.92466	.92311	.92414	.92549	.92676
55	.93150	.93006	.93099	.93221	.93337
60	.93720	.93586	.93670	.93782	[illegible]
65	.94202	.94077	.94154	.94257	.94355
70	.94616	.94498	.94569	.94665	.94756
75	.94975	.94863	.94929	.95018	.95103
80	.95288	.95183	.95245	.95328	.95408
85	.95565	.95465	.95523	.95602	.95677
90	.95812	.95716	.95771	.95845	.95916
95	.96032	.95941	.95993	.96063	.96130
100	.96230	.96143	.96192	.96259	.96322

Table I. Percentage Points of L_0

α = .05

N	2	3	4	5	6
3	.08726	.10343	.12627	.15082	.17639
4	.26961	.26141	.26769	.27759	.28911
5	.41201	.39565	.39639	.40047	.40560
6	.51144	.49221	.49098	.49336	.49662
7	.58314	.56314	.56091	.56244	.56494
8	.63689	.61706	.61429	.61528	.61730
9	.67854	.65931	.65626	.65688	.65855
10	.71171	.69325	.69007	.69044	.69185
11	.73873	.72109	.71788	.71806	.71926
12	.76115	.74433	.74113	.74118	.74222
13	.78004	.76401	.76086	.76081	.76173
14	.79617	.78089	.77780	.77768	.77849
15	.81011	.79553	.79251	.79224	[illegible]
16	.82227	.80833	.80540	.80519	.80584
17	.83297	.81964	.81679	.81654	.81713
18	.84245	.82968	.82691	.82665	.82719
19	.85092	.83867	.83598	.83570	.83620
20	.85853	.84676	.84415	.84385	.84431
21	.86540	.85408	.85155	.85124	.85166
22	.87163	.86073	.85827	.85796	.85835
23	.87732	.86680	.86441	.86410	.86446
24	.88252	.87237	.87005	.86973	.87007
25	.88730	.87748	.87523	.87491	.87523
30	.90635	.89795	.89598	.89567	.89591
35	.91990	.91257	.91082	.91052	.91072
40	.93002	.92352	.92195	.92168	.92183
45	.93787	.93204	.93062	.93036	.93049
50	.94414	.93884	.93755	.93730	.93742
55	.94926	.94441	.94322	.94299	[illegible]
60	.95352	.94905	.94795	.94774	[illegible]
65	.95712	.95298	.95195	.95175	[illegible]
70	.96020	.95634	.95538	.95519	[illegible]
75	.96287	.95926	.95836	.95817	[illegible]
80	.96520	.96181	.96096	.96078	[illegible]
85	.96726	.96406	.96325	.96309	.96314
90	.96909	.96605	.96529	.96514	.96519
95	.97073	.96784	.96712	.96700	.96702
100	.97220	.96945	.96876	.96862	.96867

Table II. Percentage Points of L_1

$\alpha = .01$

N	2	3	4	5	6
3	.06038	..09887	.12372	.15609	.18748
4	.22992	.25183	.27559	.29787	.31870
5	.37893	.39586	.41467	.43083	.44484
6	.48511	.49856	.51476	.52850	.53994
7	.56186	.57274	.58686	.59891	.60890
8	.61930	.62829	.64072	.65142	.66029
9	.66369	.67127	.68234	.69193	.69990
10	.69895	.70546	.71541	.72409	.73131
11	.72758	.73326	.74230	.75021	.75681
12	.75129	.75631	.76457	.77184	.77790
13	.77122	.77571	.78332	.79003	.79565
14	.78822	.79226	.79931	.80555	.81078
15	.80287	.80655	.81311	.81894	.82392
16	.81564	.81900	.82515	.83061	.83519
17	.82685	.82996	.83573	.84087	.84519
18	.83679	.83966	.84510	.84996	.85404
19	.84565	.84832	.85347	.85807	.86193
20	.85630	.85610	.86098	.86535	.86902
21	.86077	.86311	.86775	.87192	.87542
22	.86727	.86948	.87390	.87788	.88122
23	.87320	.87528	.87951	.88331	.88650
24	.87862	.88059	.88463	.88827	.89134
25	.88359	.88547	.88934	.89284	.89578
30	.90340	.90489	.90810	.91101	.91346
35	.91745	.91868	.92143	.92392	.92602
40	.92793	.92898	.93138	.93355	.93540
45	.93605	.93697	.93909	.94102	.94266
50	.94253	.94334	.94525	.94699	.94846
55	.94781	.94854	.95027	.95185	.95319
60	.95221	.95287	.95445	.95590	.95713
65	.95592	.95652	.95799	.95932	.96046
70	.95910	.95965	.96101	.96225	.96330
75	.96185	.96236	.96363	.96479	.96577
80	.96425	.96473	.96592	.96709	.96792
85	.96637	.96682	.96793	.96896	.96982
90	.96825	.96867	.96973	.97069	.97151
95	.96994	.97033	.97133	.97224	.97302
100	.97145	.97182	.97277	.97364	.97437

Table II. Percentage Points of L_1

$\alpha = .05$

N	2	3	4	5	6
3	.13239	.15090	.17995	.21108	.24261
4	.36242	.35480	.36258	.37372	.38636
5	.51231	.49968	.50294	.50869	.51484
6	.60749	.59386	.59555	.59976	.60428
7	.67222	.65882	.65966	.66301	.66672
8	.71885	.70609	.70644	.70919	.71235
9	.75397	.74195	.74200	.74432	.74706
10	.78134	.77006	.76992	.77192	.77434
11	.80326	.79267	.79241	.79417	.79633
12	.82120	.81125	.81091	.81247	.81442
13	.83615	.82679	.82639	.82779	.82957
14	.84880	.83996	.83593	.84080	.84244
15	.85964	.85128	.85083	.85199	.85350
16	.86903	.86111	.86064	.86171	.86312
17	.87725	.86971	.86925	.87023	.87155
18	.88450	.87732	.87685	.87777	.87900
19	.89093	.88409	.88362	.88448	.88564
20	.89670	.89015	.88968	.89049	.89158
21	.90188	.89561	.89515	.89590	.89694
22	.90657	.90055	.90010	.90081	.90180
23	.91083	.90505	.90460	.90528	.90622
24	.91472	.90916	.90872	.90936	.91026
25	.91828	.91292	.91249	.91311	.91396
30	.93241	.92788	.92749	.92798	.92868
35	.94238	.93845	.93810	.93851	.93911
40	.94978	.94632	.94600	.94635	.94687
45	.95550	.95241	.95211	.95242	.95288
50	.96005	.95725	.95698	.95726	.95767
55	.96375	.96120	.96095	.96120	.96157
60	.96683	.96449	.96425	.96448	.96482
65	.96942	.96726	.96704	.96724	.96756
70	.97164	.96963	.96942	.96961	.96990
75	.97356	.97168	.97148	.97166	.97193
80	.97524	.97347	.97328	.97345	.97370
85	.97671	.97504	.97487	.97502	.97526
90	.97802	.97644	.97628	.97642	.97665
95	.97919	.97770	.97754	.97768	.97789
100	.98025	.97882	.97867	.97880	.97900

Table III. Percentage Points of L_2

α = .01

N	2	3	4	5	6
3	.21544	.29431	.34369	.37781	.40311
4	.39811	.45597	.49383	.52038	.54021
5	.51795	.56046	.59008	.61127	.62727
6	.59948	.63211	.65614	.67363	.68695
7	.65793	.68398	.70407	.71891	.73029
8	.70170	.72316	.74037	.75323	.76315
9	.73564	.75375	.76879	.78012	.78890
10	.76270	.77830	.79161	.80174	.80961
11	.78476	.79841	.81035	.81950	.82664
12	.80309	.81519	.82601	.83435	.84087
13	.81855	.82940	.83929	.84694	.85295
14	.83176	.84158	.85068	.85776	.86333
15	.84319	.85215	.86057	.86715	.87234
16	.85317	.86139	.86924	.87538	.88023
17	.86195	.86955	.87689	.88266	.88721
18	.86975	.87680	.88369	.88913	.89342
19	.87671	.88329	.88979	.89492	.89898
20	.88297	.88913	.89527	.90013	.90399
21	.88862	.89442	.90024	.90486	.90853
22	.89376	.89922	.90476	.90916	.91266
23	.89844	.90361	.90889	.91309	.91643
24	.90273	.90763	.91267	.91669	.91989
25	.90666	.91132	.91615	.92001	.92308
30	.92239	.92612	.93009	.93328	.93583
35	.93358	.93669	.94006	.94278	.94496
40	.94195	.94461	.94754	.94991	.95182
45	.94844	.95077	.95336	.95546	.95715
50	.95363	.95570	.95802	.95991	.96142
55	.95787	.95973	.96183	.96354	.96492
60	.96140	.96309	.96501	.96658	.96784
65	.96439	.96593	.96770	.96914	.97031
70	.96694	.96836	.97000	.97134	.97242
75	.96916	.97047	.97200	.97325	.97426
80	.97109	.97232	.97375	.97492	.97586
85	.97281	.97395	.97529	.97639	.97728
90	.97432	.97540	.97666	.97770	.97854
95	.97567	.97669	.97789	.97888	.97967
100	.97689	.97786	.97900	.97993	.98069

Table III. Percentage Points of L_2

$\alpha = .05$

N	2	3	4	5	6
3	.36840	..41820	.45036	.47267	.48925
4	.54928	.57086	.58990	.60436	.61559
5	.65184	.66132	.67381	.68409	.69237
6	.71687	.72060	.72945	.73726	.74375
7	.76160	.76234	.76896	.77518	.78048
8	.79418	.79327	.79844	.80357	.80802
9	.81896	.81711	.82127	.82561	.82944
10	.83843	.83603	.83946	.84321	.84657
11	.85413	.85140	.85429	.85759	.86057
12	.86705	.86415	.86662	.86955	.87223
13	.87789	.87488	.87703	.87966	.88210
14	.88707	.88405	.88594	.88832	.89055
15	.89498	.89196	.89364	.89582	.89785
16	.90186	.89887	.90037	.90237	.90428
17	.90789	.90494	.90629	.90815	.90993
18	.91322	.91033	.91156	.91329	.91495
19	.91797	.91514	.91626	.91788	.91944
20	.92223	.91946	.92049	.92201	.92348
21	.92606	.92336	.92431	.92574	.92713
22	.92954	.92690	.92778	.92913	.93045
23	.93270	.93013	.93095	.93223	.93348
24	.93560	.93309	.93385	.93506	.93626
25	.93825	.93580	.93652	.93767	.93882
30	.94880	.94663	.94717	.94810	.94903
35	.95627	.95434	.95476	.95554	.95633
40	.96184	.96010	.96044	.96111	.96180
45	.96615	.96457	.96486	.96545	.96605
50	.96959	.96814	.96839	.96891	.96945
55	.97239	.97105	.97127	.97174	.97223
60	.97472	.97348	.97368	.97410	.97454
65	.97669	.97553	.97571	.97610	.97650
70	.97837	.97729	.97745	.97781	.97818
75	.97983	.97881	.97895	.97929	.97964
80	.98110	.98014	.98027	.98059	.98091
85	.98222	.98131	.98144	.98173	.98204
90	.98322	.98236	.98247	.98275	.98304
95	.98411	.98329	.98340	.98366	.98393
100	.98491	.98413	.98423	.98447	.98473

Table IV. Percentage Points of W_0

$\alpha = .01$

N	2	3	4	5	6
3	38.5415	49.9855	58.9218	66.3285	72.5622
4	28.6280	40.8063	51.3262	60.7529	69.3152
5	24.2086	35.3538	45.3622	54.7405	63.6253
6	21.9539	32.4086	41.8432	50.7736	59.3656
7	20.5941	30.6123	39.6598	48.2359	56.5136
8	19.6844	29.4045	38.1857	46.5107	54.5504
9	19.0330	28.5365	37.1244	45.2662	53.1295
10	18.5435	27.8823	36.3236	44.3264	52.0552
11	18.1620	27.3715	35.6978	43.5915	51.2148
12	17.8565	26.9615	35.1952	43.0011	50.5394
13	17.6062	26.6251	34.7825	42.5162	49.9845
14	17.3974	26.3441	34.4377	42.1108	49.5207
15	17.2205	26.1059	34.1452	41.7669	49.1270
16	16.0689	25.9014	33.8940	41.4715	48.7886
17	16.9373	25.7239	33.6758	41.2149	48.4950
18	16.8221	25.5683	33.4846	40.9900	48.2375
19	16.7205	25.4309	33.3157	40.7912	48.0099
20	16.6301	25.3086	33.1653	40.6143	47.8073
21	16.5491	25.1991	33.0306	40.4558	47.6257
22	16.4762	25.1005	32.9092	40.3129	47.4621
23	16.4103	25.0111	32.7993	40.1835	47.3139
24	16.3503	24.9299	32.6993	40.0658	47.1791
25	16.2955	24.8556	32.6079	39.9582	47.0558
30	16.0802	24.5635	32.2481	39.5346	46.5705
35	15.9299	24.3594	31.9967	39.2385	46.2312
40	15.8192	24.2088	31.8111	39.0199	45.9806
45	15.7341	24.0930	31.6685	38.8518	45.7880
50	15.6667	24.0013	31.5554	38.7185	45.6353
55	15.6120	23.9268	31.4636	38.6103	45.5112
60	15.5668	23.8652	31.3875	38.5207	45.4085
65	15.5287	23.8133	31.3235	38.4452	45.3220
70	15.4962	23.7690	31.2688	38.3808	45.2481
75	15.4682	23.7307	31.2217	38.3252	45.1844
80	15.4437	23.6974	31.1805	38.2767	45.1288
85	15.4220	23.6681	31.1443	38.2341	45.0799
90	15.4131	23.6420	31.1122	38.1962	45.0265
95	15.3861	23.6188	31.0836	38.1624	44.9978
100	15.3708	23.5980	31.0578	38.1321	44.9630

Table IV. Percentage Points of W_0

α = .05

N	2	3	4	5	6
3	29.2635	40.8391	49.6644	56.7496	62.4613
4	20.9726	32.2001	42.1739	51.2641	59.5659
5	17.1743	27.8168	37.0143	45.7554	54.1430
6	16.0926	25.5186	34.1449	42.3913	50.3931
7	15.1011	24.1174	32.3789	40.2835	47.9673
8	14.4371	23.1736	31.1861	38.8544	46.3109
9	13.9611	22.4943	30.3261	37.8228	45.1132
10	13.6032	21.9818	29.6764	37.0430	44.2071
11	13.3242	21.5812	29.1682	36.4327	43.4978
12	13.1007	21.2596	28.7598	35.9419	42.9272
13	12.9174	20.9955	28.4243	35.5387	42.4583
14	12.7646	20.7748	28.1438	35.2015	42.0661
15	12.6351	20.5877	27.9058	34.9153	41.7332
16	12.5240	20.4269	27.7012	34.6693	41.4470
17	12.4277	20.2874	27.5236	34.5556	41.1984
18	12.3433	20.1651	27.3678	34.2682	40.9804
19	12.2688	20.0570	27.2302	34.1025	40.7876
20	12.2025	19.9608	27.1077	33.9551	40.6160
21	12.1432	19.8747	26.9979	33.8230	40.4622
22	12.0898	19.7970	26.8989	33.7038	40.3236
23	12.0415	19.7267	26.8093	33.5960	40.1980
24	11.9975	19.6628	26.7278	33.4978	40.0837
25	11.9574	19.6043	26.6532	33.4080	39.9792
30	11.7995	19.3743	26.3598	33.0546	39.5678
35	11.6894	19.2136	26.1547	32.8075	39.2800
40	11.6081	19.0949	26.0132	32.6245	39.0675
45	11.5458	19.0038	25.8867	32.4846	38.9040
50	11.4963	18.9315	25.7944	32.3734	38.7743
55	11.4562	18.8728	25.7194	32.2830	38.6691
60	11.4230	18.8242	25.6573	32.2081	38.5819
65	11.3951	18.7833	25.6050	32.1451	38.5095
70	11.3713	18.7484	25.5604	32.0913	38.4458
75	11.3507	18.7182	25.5218	32.0448	38.3917
80	11.3328	18.6920	25.4882	32.0043	38.3445
85	11.3170	18.6688	25.4586	31.9686	38.3029
90	11.3030	18.6483	25.4324	31.9370	38.2661
95	11.2905	18.6300	25.4090	31.9088	38.2332
100	11.2793	18.6136	25.3880	31.8834	38.2037

Table V. Percentage Points of W_1

$\alpha = .01$

N	2	3	4	5	6
3	33.6859	43.1664	50.1539	55.7192	60.2661
4	23.5203	33.0961	41.2425	48.4437	54.8877
5	19.4082	27.8010	35.2112	42.1026	48.6029
6	17.3612	25.0573	31.8742	38.2630	44.3740
7	16.1418	23.4079	29.8463	35.8849	41.6729
8	15.3331	22.3085	28.4903	34.2881	39.8467
9	14.7576	21.5233	27.5201	33.1441	38.5358
10	14.3272	20.9345	26.7916	32.2842	37.5498
11	13.9931	20.4765	26.2244	31.6145	36.7814
12	13.7264	20.1101	25.7703	31.0779	36.1656
13	13.5084	19.8104	25.3985	30.6385	35.6612
14	13.3270	19.5606	25.0885	30.2719	35.2403
15	13.1737	19.3492	24.8261	29.9615	34.8838
16	13.0423	19.1680	24.6010	29.6953	34.5779
17	12.9286	19.0110	24.4059	29.4644	34.3127
18	12.8292	18.8736	24.2351	29.2622	34.0804
19	12.7415	18.7524	24.0844	29.0838	33.8754
20	12.6635	18.6446	23.9503	28.9251	33.6930
21	12.5939	18.5482	23.8304	28.7831	33.5298
22	12.5312	18.4614	23.7224	28.6552	33.3828
23	12.4745	18.3829	23.6247	28.5395	33.2498
24	12.4230	18.3115	23.5359	28.4342	33.1288
25	12.3759	18.2463	23.4547	28.3381	33.0183
30	12.1913	17.9902	23.1358	27.9603	32.5838
35	12.0628	17.8117	22.9134	27.6967	32.2807
40	11.9681	17.6802	22.7495	27.5023	32.0571
45	11.8955	17.5792	22.6236	27.3531	31.8855
50	11.8381	17.4993	22.5240	27.2350	31.7495
55	11.7915	17.4345	22.4431	27.1391	31.6392
60	11.7529	17.3808	22.3762	27.0597	31.5478
65	11.7205	17.3357	22.3199	26.9929	31.4710
70	11.6928	17.2971	22.2718	26.9359	31.4054
75	11.6690	17.2639	22.2304	26.8867	31.3488
80	11.6482	17.2349	22.1942	26.8438	31.2994
85	11.6299	17.2095	22.1624	26.8061	31.2560
90	11.6137	17.1869	22.1342	26.7726	31.2175
95	11.5992	17.1667	22.1090	26.7428	31.1832
100	11.5862	17.1486	22.0864	26.7160	31.1523

Table V. Percentage Points of W_1

$\alpha = .05$

N	2	3	4	5	6
3	24.2640	34.0409	41.1618	46.6659	50.9873
4	16.2394	24.8688	32.4639	39.3697	45.6477
5	13.3765	20.8134	27.4916	33.7962	39.8344
6	11.9622	18.7598	24.8772	30.6733	36.2679
7	11.1208	17.5269	23.2976	28.7676	34.0524
8	10.5631	16.7049	22.2415	27.4903	32.5622
9	10.1663	16.1178	21.4856	26.5751	31.4932
10	9.8696	15.6773	20.9179	25.8871	30.6890
11	9.6394	15.3347	20.4758	25.3510	30.0621
12	9.4555	15.0606	20.1217	24.9214	29.5597
13	9.3053	14.8363	19.8318	24.5695	29.1480
14	9.1803	14.6494	19.5900	24.2760	28.8044
15	9.0746	14.4912	19.3853	24.0274	28.5134
16	8.9841	14.3556	19.2098	23.8141	28.2637
17	8.9058	14.2381	19.0576	23.6291	28.0471
18	8.8373	14.1353	18.9243	23.4672	27.8575
19	8.7768	14.0445	18.8067	23.3242	27.6900
20	8.7232	13.9638	18.7021	23.1971	27.5411
21	8.6752	13.8917	18.6085	23.0832	27.4078
22	8.6320	13.8267	18.5243	22.9808	27.2877
23	8.5929	13.7679	18.4480	22.8880	27.1790
24	8.5574	13.7145	18.3787	22.8037	27.0802
25	8.5250	13.6657	18.3154	22.7267	26.9899
30	8.3978	13.4739	18.0665	22.4238	26.6350
35	8.3093	13.3403	17.8929	22.2125	26.3874
40	8.2441	13.2418	17.7649	22.0567	26.2047
45	8.1940	13.1662	17.6667	21.9371	26.0645
50	8.1545	13.1064	17.5889	21.8424	25.9534
55	8.1224	13.0578	17.5257	21.7655	25.8632
60	8.0958	13.0176	17.4735	21.7018	25.7885
65	8.0735	12.9838	17.4295	21.6483	25.7257
70	8.0544	12.9550	17.3920	21.6026	25.6721
75	8.0380	12.9301	17.3596	21.5631	25.6257
80	8.0237	12.9084	17.3314	21.5287	25.5855
85	8.0111	12.8893	17.3066	21.4985	25.5500
90	7.9999	12.8724	17.2846	21.4717	25.5186
95	7.9899	12.8573	17.2649	21.4477	25.4905
100	7.9810	12.8438	17.2473	21.4262	25.4653

Vita

Arthur J. Sherwood was born 26 November 1947 in Muskegon, Michigan. He enlisted in the USAF in January 1969 and after completion of technical school was awarded the AFSC 301X1, Aircraft Electronic Equipment Repairman, and served in this capacity at the 410th Bombardment Wing, K. I. Sawyer AFB, Michigan. He was selected for AECP and upon graduation from Oklahoma State University in December 1972 was commissioned in the USAF through OTS, April 1973. He was awarded an AFSC 3031, Communications Maintenance Officer, and was stationed at Grand Forks AFB, North Dakota until March of 1975. He was selected for Navigation Training at Mather AFB and received his wings in February 1976. He then served as a KC-135A Navigator and Instructor Navigator until entering the School of Engineering, Air Force Institute of Technology, in September 1981.

Permanent Address: 293 W. Norton Avenue
Norton Shores, Michigan 49444

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE — READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER AFIT/GAM/MA/83M-1	2. GOVT ACCESSION NO. AD-A127358	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) ON THE DISTRIBUTION OF THE LIKELIHOOD RATIO CRITERIA ASSOCIATED WITH K SAMPLES OF TWO CORRELATED RANDOM VARIABLES		5. TYPE OF REPORT & PERIOD COVERED M.S. Thesis 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Arthur J. Sherwood, Captain, USAF		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Air Force Institute of Technology (AFIT/EN) Wright-Patterson AFB OH 45433		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS		12. REPORT DATE March 1983 13. NUMBER OF PAGES 85
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

Approved for public release: IAW AFR 190-17.
Lynn E. Wolaver
LYNN E. WOLAVER
Dean for Research and Professional Development
Air Force Institute of Technology (ATC)
Wright-Patterson AFB OH 45433

7 APR 1983

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Likelihood Ratio
Hypotheses
Wishart Matrix
Asymptotic
Distribution Function

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Let a random sample of size N_g be drawn from a p-variate normal population $N_p(\underline{\mu}^g, \underline{\Sigma}_g)$ $g = (1,2,\ldots,k)$. In this thesis we consider the problem of testing the following hypotheses:

DD FORM 1473, 1 JAN 73 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED
SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

[i] H_0: $\underline{\mu}^1 = \underline{\mu}^2 = \ldots = \underline{\mu}^k$,

$\underline{\Sigma}_1 = \underline{\Sigma}_2 = \ldots = \underline{\Sigma}_k$

[ii] H_1: $\underline{\Sigma}_1 = \underline{\Sigma}_2 = \ldots = \underline{\Sigma}_k$. The means can be any value

[iii] H_2: $\underline{\mu}^1 = \underline{\mu}^2 = \underline{\mu}^k$ given $\underline{\Sigma}_1 = \underline{\Sigma}_2 = \ldots \underline{\Sigma}_k$

against the general alternatives.

Likelihood ratio criteria and their sampling distributions are derived for p = 2 and equal sample sizes. From these distributions, tables of percentage points for the three likelihood ratio criteria are computed.

A useful approximation is also obtained. The theoretical results are then applied to actual data.

DATE
ILME